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Round-robin Tournaments with Minimum Number of Breaks and Two Teams per Club

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Abstract

We study round-robin tournaments for $2n$ teams. Here n is either interpreted as the number of clubs, each having two teams, or the number of strength groups with two teams each. For even n we give a construction of a single round-robin tournament for $2n$ teams with $2n-2$ breaks, where the teams of the same club have complementary home-away patterns and play against each other in the first round. If the pairs of teams are strength groups, then a cyclic permutation of the constructed schedule results in a group-balanced tournament.

Keywords: Sports, Round-Robin Tournaments, Break, Strength Group.

1 Introduction and Problem Specification

We consider sports leagues having a set of $2n$ teams. A single round robin tournament (SRRT) is a schedule of matches between pairs of teams such that each pair of teams meets exactly once and each team plays exactly once in each of $2n-1$ rounds. For SRRTs a well-known construction exists, see [5, 6]; this construction is sometimes called the Circle Method. Here we present this method in algebraic form.

Circle Method

(a) For $i, j < 2n$ and $i \neq j$, the teams i and j play in round k if $i+j-1 = k \mod (2n-1)$.

(b) For $i < 2n$, the teams i and $2n$ play in round k if $2i-1 = k \mod (2n-1)$.

Each match is carried out at one of both opponents' venues, which is why breaks come into play: if a team i plays twice at home or twice away in rounds k and $k+1$, then we say that i has a *break* in round $k+1$. The decision on the home team for each match is called a *home-away assignment*. Each home-away assignment in a single round-robin tournament for $2n$ teams yields at least $2n-2$ breaks, and a home-away assignment with exactly $2n-2$ breaks exists for the schedule constructed using the Circle Method; see again [5, 6]. A home-away assignment leads to a *home-away pattern* for each team, that is, a string of length $2n-1$ specifiying for each round whether the team plays at home or away. Two home-away patterns that are different in each round are called *complementary*. We say that two teams *play complementary* if their home-away patterns are complementary.

SRRTs with particular structures have been adressed in several pieces of work. In [1, 4, 5, 7] SRRTs with a minimum number of breaks are considered. Many results of these papers are valid for equitable SRRTs, that is, SRRTs where each team has exactly one break. Note that the only difference is that two teams have breaks in the first round (looking at rounds in a circular way) in an SRRT having a minimum number of breaks, while in an equitable SRRT no team has a break in the first round. We will use this method of cyclic permutations in Section 3.

We consider the problem of n clubs having two teams and one venue each. Two teams belonging to the same club share this single venue, which means that at most one team of a club can play at home in each round. Moreover, a minimum number of breaks is required and for fairness reasons the matches between teams of the same club must be arranged in the first round. It is known that in SRRTs with $2n$ teams and the minimum number of breaks there are n pairs of complementary home-away patterns. Suppose teams x and y play for the same club. If team x has a home break in round $k+1$ (i.e., plays at home in rounds k and $k+1$), then team y must play away in rounds k and $k+1$ in order to satisfy the venue restriction. Since both x and y have exactly one break, this implies that x and y have complementary home-away patterns. In general this means that teams of the same club play complementary.

Hence we consider SRRTs with the following properties:

Property I. The number of breaks equals $2n-2$.

Property II. The teams of the same club play complementary.

Property III. The teams of the same club meet in the first round.

The contributions of this paper are twofold. Firstly, we provide a construction scheme for SRRTs having Properties I, II, and III for all even n; see Section 2. It is well-known ([5, 6]) that Property I implies Property II, however not Property III; it is remarkable that with adding this property we seem to loose all odd n. The second contribution is the construction of group-balanced SRRTs with $2n-2$ breaks for n groups of size 2, for all even n; this is presented in Section 3. Section 4 contains the conclusions and mentions some open problems.

2 Sports schedules with multiple teams per club

The purpose of this section is to prove the existence of an SRRT satisfying Properties I, II, and III for even n. We do this by an explicit construction closely resembling the Circle Method. However, where the Circle Method has only one exceptional team (namely team $2n$, sometimes called team ∞ in the literature), our construction (the Adapted Circle Method) will have two exceptional teams, namely teams $2n-1$ and $2n$.

Thoughout this section we consider the case of n clubs (n even) and $2n$ teams, numbered from 1 to $2n$. We number the teams in such a way that for $i < n$ the teams i and $i+n-1$ belong to the same club, as well as the teams $2n-1$ and $2n$.

Adapted Circle Method

(a) In round 1, each team plays against the team of the same club.

(b) For $i, j \leq 2n-2$, the teams i and j of different clubs play in round $k \geq 2$ if $i+j = k \mod (2n-2)$.

Note that in rule (b) there is a shift of one round as compared to rule (a) of the Circle Method, due to the special first round. It remains to describe the matches involving the teams $2n-1$ and $2n$. For team i these must be played in the rounds k, where either $k = 2i \mod (2n-2)$ or $k = 2i+n-1 \mod (2n-2)$.

(c1) For $1 \leq i \leq \frac{1}{2}n$, team i plays against $\begin{cases} 2n-1 & \text{in round } 2i+n-1 \\ 2n & \text{in round } 2i\ . \end{cases}$

(c2) For $\frac{1}{2}n+1 \leq i \leq n-1$, team i plays against $\begin{cases} 2n-1 & \text{in round } 2i \\ 2n & \text{in round } 2i-(n-1) \ . \end{cases}$

(c3) For $n \leq i \leq \frac{3}{2}n-1$, team i plays against $\begin{cases} 2n-1 & \text{in round } 2i-2(n-1) \\ 2n & \text{in round } 2i-(n-1) \ . \end{cases}$

(c4) For $\frac{3}{2}n \leq i \leq 2n-2$, team i plays against $\begin{cases} 2n-1 & \text{in round } 2i-3(n-1) \\ 2n & \text{in round } 2i-2(n-1) \ . \end{cases}$

(d) The opponents of the teams $2n-1$ and $2n$ are those implied by rule (c).

It is not difficult to verify that every team i plays against each of the $2n-1$ other teams for $i \leq 2n-2$. For the teams $2n-1$ and $2n$ the opponents follow from rule (d). Team $2n-1$ plays against team $2n$ in round 1, and plays against the teams $1, 2, \ldots, 2n-2$ in rounds

$$\begin{gathered} n+1, n+3, \ldots, 2n-3, 2n-1, \quad n+2, n+4, \ldots, 2n-4, 2n-2, \\ 2, 4, \ldots, n-2, n, \quad 3, 5, \ldots, n-3, n-1, \end{gathered} \tag{1}$$

respectively. For team $2n$ the analysis is similar. This construction fails if n is odd, since in that case the rounds in (1) are not all different.

To define the home-away assignment satisfying Properties I and II we introduce the *band*.

Definition 1. *Consider the opponent schedule constructed using the Adapted Circle Method. The* band B *consist of all pairs* (i, k) *of team* $i \leq 2n-2$ *and round* k*, for which*

$$\frac{1}{2}(k+\delta_k) \leq i < \frac{1}{2}(k+\delta_k)+n-1 \ ,$$

where

$$\delta_k = \begin{cases} 1 & \textit{if } k \leq n \\ 0 & \textit{if } k > n \ . \end{cases}$$

Using the band, we construct a home-away assignment as follows.

Lemma 1. *Given the opponent schedule constructed using the Adapted Circle Method, a home-away assignment is obtained as follows. For* $i \leq 2n-2$*, team* i *plays home in round* k *if and only if one of the following holds:*

$$\begin{cases} (i,k) \in B & \textit{and } k \textit{ is odd} \\ (i,k) \notin B & \textit{and } k \textit{ is even} \ . \end{cases}$$

In round 1 team $2n-1$ *plays home against team* $2n$*. In every other round the teams* $2n-1$ *and* $2n$ *play complementary to their opponents.*

Proof. It is clear that each team i plays either home or away in each round. Hence, in order to prove that Lemma 1 defines a proper home-away assignment, it suffices to prove that if two teams meet in round k, then one plays home and the other plays away. For all matches involving the teams $2n-1$ and $2n$, this immediately follows from the formulation of the lemma. So suppose now that the teams i and j meet in round k, and assume without loss

	1	2	3	4	5	6	7	8	9	10	11
1	6	-12	-2	3	-4	5	-11	7	-8	9	-10
2	7	-10	1	-12	-3	4	-5	6	-11	8	-9
3	8	-9	10	-1	2	-12	-4	5	-6	7	-11
4	9	-8	12	-10	1	-2	3	11	-5	6	-7
5	10	-7	8	-9	12	-1	2	-3	4	11	-6
6	-1	11	7	-8	9	-10	12	-2	3	-4	5
7	-2	5	-6	11	8	-9	10	-1	12	-3	4
8	-3	4	-5	6	-7	11	9	-10	1	-2	12
9	-4	3	-11	5	-6	7	-8	-12	10	-1	2
10	-5	2	-3	4	-11	6	-7	8	-9	-12	1
11	12	-6	9	-7	10	-8	1	-4	2	-5	3
12	-11	1	-4	2	-5	3	-6	9	-7	10	-8

Table 1. The opponent schedule for 12 teams with the band in grey.

of generality that $i < j \leq 2n-2$. We have to prove that exactly one of the pairs (i,k) and (j,k) is in the band. For $k=1$ this is correct, since $j = i+n-1$, which implies that $(i,1)$ is in the band, and $(j,1)$ is outside. For $k>1$ the opponents follow rule (b) of the Adapted Circle Method, implying that either $i+j=k$ or $i+j=k+2n-2$.

In the first case $i < \frac{1}{2}k$, so $(i,k) \notin B$. On the other hand $j > \frac{1}{2}k$ implies that $\frac{1}{2}(k+\delta_k) \leq j$. To prove that $(j,k) \in B$, it remains to prove that $j < \frac{1}{2}(k+\delta_k)+n-1$. For this we note that $j<k$ and $k \leq 2n-1$ imply

$$j \leq k-1 = \frac{1}{2}k + \frac{1}{2}k - 1 \leq \frac{1}{2}k + \frac{1}{2}(2n-1) - 1 = \frac{1}{2}k + n - \frac{3}{2} < \frac{1}{2}(k+\delta_k)+n-1.$$

Hence we have that $j < \frac{1}{2}(k+\delta_k)+n-1$ and we proved that $(j,k) \in B$.

In the second case $j > \frac{1}{2}k+n-1$, which implies that $j \geq \frac{1}{2}(k+\delta_k)+n-1$, and hence $(j,k) \notin B$. At the same time we have that $i < \frac{1}{2}k+n-1$, so surely $i < \frac{1}{2}(k+\delta_k)+n-1$. To prove that $(i,k) \in B$ it remains to prove that $i \geq \frac{1}{2}(k+\delta_k)$. To get a contradiction we assume that $i < \frac{1}{2}(k+\delta_k)$. This implies that $i \leq \frac{1}{2}k$. If $i = \frac{1}{2}k$ (implying that k is even), team i plays against team $2n-1$ or $2n$; cases we already discarded. If $i < \frac{1}{2}k$ then the opponent is $k-i$, implying that $j<k$. This is in contradiction with $j > \frac{1}{2}k+n-1$. Hence $(i,k) \in B$. □

Table 1 contains the result of the construction of an opponent schedule for 12 teams using the Adapted Circle Method. The entries belonging to the band are colored grey, and a positive (respectively negative) entry in row i and column k indicates that team i plays home (respectively away) in round k. For other values of n a java script is available, see http://wwwhome.math.utwente.nl/ ~postgf/RoundRobinWithTwoTeamsPerClub.html.

Using the Adapted Circle Method and the home-away assignment in Lemma 1 we are able to prove our main result.

Theorem 1. *(Main Theorem)*
For even n there exists a single round robin tournament for $2n$ teams, satisfying Properties I, II, and III in Section 1.

Proof. (Main Theorem) To prove Property I we note that team $i \leq 2n-2$ does *not* have a break in round k, $2 \leq k \leq 2n$, if and only if the pairs $(i, k-1)$ and (i,k) are both inside or both outside the band. Since for each $i \leq 2n-2$ the transition from inside to outside the band happens exactly once, each of those teams have exactly 1 break. For every round k, if team $2n-1$ plays against team i, then the pair (i,k) is outside B. Hence the home-away pattern for team $2n-1$ is home–away–home– $\cdots$ –away–home, which means that team $2n-1$ does not have a break. Similarly, the pair (i,k) corresponding to the opponent i of team $2n$ in round k, is always inside B. Hence the home-away pattern of team $2n$ is complementary to that of team $2n-1$, which means that team $2n$ does not have a break either.

To prove Property II we note that for every fixed round exactly $n-1$ "consecutive" teams belong to the band. Hence for $i < n$ the teams i and $i+n-1$ play complementary, since for each round k either (i,k) or $(i+n-1,k)$ is inside the band. As noted in the first part of the proof the teams $2n-1$ and $2n$ play complementary as well.

Property III follows directly from the construction of the first round. □

3 Group-balanced schedules

The schedule constructed in Section 2 is much more structured than required by Properties I, II, and III. In fact the following is true:

> **Property IV**. If a team of club A plays against a team of club B, then the other teams of the clubs A and B meet in the same round.

Looking at the rounds 2 to $2n-1$ at *club* level, which can be done due to Property IV, we see that the clubs play exactly a *double* round-robin tournament in these $2n-2$ rounds; moreover the two matches between the same clubs are exactly $n-1$ rounds apart. This extra structure can be used to construct a group-balanced tournament for strength groups of size 2, which we describe now.

In [2, 3] fair SRRTs with regard to strength groups have been considered. Here we consider the special case where the set of teams is partitioned into n strength groups of size 2. The SRRT is called *group-balanced* if a team plays against distinct teams j and j' from the same strength group in two rounds having absolute difference exactly n; the teams in the same group meet in round n. In our case the pairs of teams of the same club mentioned in Section 2 will form the strength groups. We modify the schedule obtained using the Adapted Circle Method by rotating every round $n-1$ "slots" to the right, that is, round k becomes round $k+n-2 \bmod (2n-2)+1$. In particular the first round is moved to round n. Table 2 provides the schedule obtained in this way from the schedule in Table 1.

Firstly, we observe that the obtained schedule is group-balanced. As we noted before the Adapted Circle Method yields a schedule such that for each strength group the matches against the two teams in another strength group are carried out in two rounds that differ $n-1$. Hence, after rotating these matches are played in rounds with difference exactly n. Thus, the schedule is group-balanced.

Secondly, we show that the number of breaks is $2n-2$. Note that, in an SRRT with a minimum number of breaks, for two teams the last entry of the home-away pattern equals the

	1	2	3	4	5	6	7	8	9	10	11
1	-11	7	-8	9	-10	6	-12	-2	3	-4	5
2	-5	6	-11	8	-9	7	-10	1	-12	-3	4
3	-4	5	-6	7	-11	8	-9	10	-1	2	-12
4	3	11	-5	6	-7	9	-8	12	-10	1	-2
5	2	-3	4	11	-6	10	-7	8	-9	12	-1
6	12	-2	3	-4	5	-1	11	7	-8	9	-10
7	10	-1	12	-3	4	-2	5	-6	11	8	-9
8	9	-10	1	-2	12	-3	4	-5	6	-7	11
9	-8	-12	10	-1	2	-4	3	-11	5	-6	7
10	-7	8	-9	-12	1	-5	2	-3	4	-11	6
11	1	-4	2	-5	3	12	-6	9	-7	10	-8
12	-6	9	-7	10	-8	-11	1	-4	2	-5	3

Table 2. The group-balanced schedule for 12 teams with the band in gray.

first entry (home-home or away-away). Hence we could say that these teams have a break in the first round and, then, each team has exactly one break. Note that by rotating rounds we do not destroy this property. Hence, the modified schedule has a minimum number of breaks if and only if two teams have a break in the first period. The Adapted Circle Method yields an opponent schedule such that teams $\frac{1}{2}n$ and $\frac{3}{2}n-1$ have a break in round $n+1$, which becomes round 1 after rotating. Thus, the obtained schedule has the minimum number of breaks. In Table 2 we can see that for the teams $\frac{1}{2}n$ and $\frac{3}{2}n-1$ either the first or the last match lies in the band, which illustrates their breaks in the first round.

4 Conclusions

We presented a construction for an SRRT with the minimum number of breaks where teams of the same club meet in the first round, and have complementary home-away assignments. The construction only works for an even number of clubs. Formulating the problem as an ILP model and solving for small instances ($n=3$ and $n=5$) suggests that for an odd number of clubs such a construction is not possible. An interesting question is what the minimum number of breaks is in this case, and how to construct such schedules.

Although we do not know whether an SRRT with properties I to III exists for some odd n larger than 5 we can easily see that no SRRT with properties I to IV can exist for odd n. In fact, there is no SRRT with properties III and IV. Due to property IV clubs must be grouped in pairs in each round apart from the round reserved for inner club matches. In each of those rounds, this leaves a single club if the number of clubs is odd. Hence the two teams of this club must play against each other in a round where no other inner club matches are carried out.

The combination of break minimization and fairness with regard to strength groups raises interesting open questions for future research:

- What is the minimum number of breaks in a group-balanced SRRT with $2n$ teams and g strength groups?

- What is the computational complexity of the break minimization problem for a group-balanced SRRT?

Our result in Section 3 gives an answer to the first question for the special case of strength groups of size 2.

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