

Market Efficiency and Entropy of Foreign Exchange Markets and Stock Markets

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Abstract

In this work we apply two different methods to estimate the relative entropy of foreign exchange markets and stock markets. The first method uses relative entropy with the volatility probability distribution obtained from the solution of the Fokker-Planck equation of the Heston model corresponding to the financial information in each case [1]. The second method is the approximate entropy proposed by Pincus *et al* [2]. The results were compared with standard measures of weak efficiency such as the Geary test and the autocorrelation function. The relative entropy is used as a measure of stability and maturity of financial markets from financial information of some considered emerging markets (Argentina, Brazil, Colombia, Mexico) and some considered mature markets (England, Switzerland, Sweden, United States). This work gives new information based on more fundamental methods to compare financial markets in contrast with the traditional random and independence criteria used to measure efficiency and stability. This work gives a comparative approach between traditional methods and methods from statistical physics and information theory to analyze financial time series that may give new valuable and useful information of the financial markets dynamics.

1 Introduction

The efficiency of the economical markets is a changing problem where concepts and methods of statistical physics have been increasingly applied with interesting results compared with those of other methods traditionally used in economical and financial analysis. The work of Eugene Fama is a reference work using the Geary test and correlation functions to estimate the weak efficiency of markets [3]. Recent works have used methods from statistical physics and nonlinear dynamics to estimate the weak efficiency of markets from empirical information, like the approximated entropy developed by Pincus [2]. In this work we use the relative entropy method [6] to measure the weak efficiency of exchange and

capital markets of different American and European countries, and compare the results with those obtained with the methods used by Fama and Pincus. The time period of study is between 2001-07-03 and 2008-07-03 of daily data corresponding to approximately 1750 data points for each country. The information of the stock markets was obtained from Yahoo.com [4] and the exchange indexes were obtained from the central bank of the corresponding country. Annexed we present the plots of this financial information where the data was rescaled by subtracting the value at time zero to all the data points of the corresponding time series. It is clearly observed that the exchange and stock markets follow different dynamics and their weak efficiency seems to be different depending on the method used to measure the efficiency. In addition the degree of weak efficiency is not always consistent with the economy of the corresponding country as they are traditionally measured.

2 Relative Entropy Method

The method of relative entropy generates a measure, D , of the difference between two probability distributions [7]. If $Q(x)$ represents the probability distribution of the historical returns x , given by $x_t = \log(S_t/S_{t-1})$, where S_t are the corresponding prices of stock markets or foreign exchange market. D is a measure of the weak efficiency of the corresponding market represented by the probability distribution $P(x)$ of its returns x . The relative entropy D is given by the following equation from information theory,

$$D = - \int_{-\infty}^{\infty} P(x) \log \frac{P(x)}{Q(x)} dx. \quad (1)$$

The results of D depend on the probability distributions P and Q . In this work we estimate P by the method developed by Dragulescu for small periods of time using the Heston model [1], which is consistent with the information used in this work corresponding to daily prices. Considering the traditional weak efficiency hypothesis, where the efficiency of a market depends on the randomness of the prices, for an ideal market the prices must follow a random walk. Therefore, the probability distribution of the prices of an ideal market Q , must be a normal distribution [8]. With these considerations, equation (1) gives small values tending to zero as the market represented by the distribution P is more efficient under the weak efficiency hypothesis mentioned above. For this methodology, it is very important estimate the distribution P from the available information. In this work we estimate P by modeling the dynamics of the prices (returns) by a geometric Brownian dynamics with stochastic volatility and reversion to the mean value [8]. This model is given by

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \quad (2a)$$

$$dv_t = -\gamma(v_t - \theta)dt + \kappa \sqrt{v_t} dW_t^{(2)}, \quad (2b)$$

where μ is the tendency to increase, γ is the relaxation time to the mean value, θ is the long term mean value of the volatility, and κ is the noise parameter of the variance. In this equation the W_i represents

a standard and independent Wiener process. In order to guaranty the stationarity of the process, the prices S_t are transformed into logarithmic returns r_t , by $r_t = \log(S_t/S_{t-1})$. To obtain the stochastic equation of the returns, the Ito's lemma is applied to equation (2a):

$$dr_t = \left(\mu - \frac{v_t}{2} \right) dt + \sqrt{v_t} dW_t^{(1)}, \quad (3)$$

with the change of variable $x_t = r_t - \mu t$, equation (4) becomes,

$$dx_t = -\frac{v_t}{2} dt + \sqrt{v_t} dW_t^{(1)}. \quad (4)$$

The results of this procedure gives the Heston model:

$$\begin{aligned} dr_t &= \left(\mu - \frac{v_t}{2} \right) dt + \sqrt{v_t} dW_t^{(1)}, \\ dv_t &= -\gamma(v_t - \theta)dt + \kappa\sqrt{v_t} dW_t^{(2)}. \end{aligned} \quad (5)$$

The corresponding probability distribution obtained by Dragulescu [1] for short time periods is:

$$P(x) = \frac{2^{1-\alpha} e^{-x/2}}{\Gamma(\alpha)} \sqrt{\frac{\alpha}{\pi\theta}} y^{\alpha-1/2} K_{\alpha-1/2}(y) \quad (6)$$

where $y = \sqrt{\frac{2\alpha x^2}{\theta t}}$, $\alpha = \frac{2\gamma\theta}{\kappa^2}$, $\Gamma(\alpha)$ is the gamma function and K_c is the modified Bessel function of order c . It is important to remember that this probability distribution corresponds to a model with stochastic volatility. In contrast, to estimate the probability distribution Q of the reference ideal market, we use a geometric Brownian motion with constant volatility σ . Applying the Itos's lemma in equation (7) we obtain the process for returns 8.

Stock Markets			Foreign Exchange Markets		
Country	D	Orden	Country	D	Orden
Colombia	0.0811	1	Colombia	1.4248	2
Brazil	0.0358	7	Brazil	0.1833	6
Argentina	0.0532	4	Argentina	1.5020	1
Mexico	0.0331	8	Mexico	0.6293	4
United States	0.0611	2	United States	NA	NA
Euro	NA	NA	Euro	0.1600	7
England	0.0459	5	England	0.0709	8
Sweden	0.0544	3	Sweden	1.3936	3
Switzerland	0.0452	6	Switzerland	0.2411	5

Table 1: Relative Entropy: Left stocks Markets Right Foreigns Exchangue markets

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{(1)}, \quad (7)$$

$$dr_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t. \quad (8)$$

With the change of variable $x_t = r_t - \mu t$ we get

$$dx_t = -\frac{\sigma^2}{2}dt + \sigma dW_t. \quad (9)$$

with a normal probability distribution:

$$Q(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x+\frac{\sigma^2}{2})^2}{2\sigma^2}} \quad (10)$$

The results of D for the eight countries and their corresponding stock and exchange markets are presented in table 1.

3 Other methods

Pincus Entropy

We follow to Gabjin in this method [5]. The approximated entropy $ApEn$ was proposed by Pincus with the purpose of quantify the randomness of a time series [2]. $ApEn$ is given by

$$ApEn(m, r) = \Phi^{m(r)} - \Phi^{m+1(r)}, \quad (11)$$

where m is the embedding dimension, r the tolerance in similarity. The function $\Phi^{m(r)}$ is given by

$$\Phi^{m(r)} = \frac{\sum_{t=1}^{N-m+1} \ln |C_t^m(r)|}{(N-m+1)-1} \quad (12)$$

with

$$C_t^m(r) = \frac{B_i(r)}{(N-m+1)} \quad (13)$$

where $B_t(r)$ Is the number of data points pairs with a distance between them equal or small to r

$$B_i \equiv d[x(t), x(t')] \leq r \quad (14)$$

the distance $d[x(t), x(t')]$ between two vectors $x(t)$ y $x(t')$ in \mathbb{R}^m is defined for

$$d[x(t), x(t')] = \max_{k=1,2,\dots,m} (|u(t+k-1) - u(t'+k-1)|) \quad (15)$$

where $u(k)$ is a time series. The $ApEn$ value compares the relative magnitude between repeated pattern occurrences for the embedding dimensions, m and $m+1$. When the time series data have a high degree of randomness, the $ApEn$ is large. On the other hand, $ApEn$ is small for the time series with a low degree of randomness [5]. Therefore, $ApEn$ can be use as a measure of weak efficiency. The results presented in table 2 were obtained with $m = 2$ and $r = 20\%$ of the standard deviation as is generally recommended [2].

Stock Markets			Foreign Exchange Markets		
Country	<i>ApEn</i>	Orden	Country	<i>ApEn</i>	Orden
Colombia	1.59	1	Colombia	1.39	2
Brazil	1.77	8	Brazil	1.50	3
Argentina	1.65	4	Argentina	0.87	1
Mexico	1.73	7	Mexico	1.77	4
United States	1.68	5	United States	NA	NA
Euro	NA	NA	Euro	1.79	6
England	1.61	2	England	1.80	7
Sweden	1.69	6	Sweden	1.82	8
Switzerland	1.64	3	Switzerland	1.78	5

Table 2: *ApEn*: Left, stocks Markets. Right, Foreigns Exchange markets

Geary Test

The Geary test is a nonparametric test used by Fama in his pioneering work of time series analysis of financial markets [13]. A run is defined as a sequence of price changes of the same sign. For example, a plus run of length i is a sequence of i consecutive positive price changes preceded and followed by negative change of price. If it is assumed that the sample proportions of positive/negative price changes are good estimates of the population proportions [3], the number of sequences are a measure of positive correlation and therefore, smaller weak efficiency.

If $N = N_1 + N_2$ is the number of data points with $N_1 =$ number of positive returns and $N_2 =$ number of negative returns, and R is the number of sequences, under the null hypothesis of independent returns, and assuming that N_1 and N_2 are both greater than 10, the number of sequences is (asymptotically) normally distributed with mean value $E(R)$,

$$E(R) = \frac{2N_1N_2}{N} + 1 \quad (16)$$

and variance $E(r)$.

$$\sigma_R^2 = \frac{2N_1N_2(2N_1N_2 - N)}{(N)^2(N - 1)} \quad (17)$$

In order to quantify the results of this method we use the measure $d_g = |R - E(R)|$. When d_g is small tending to zero, the returns are more random and therefore, the corresponding market is more efficient in terms of the mentioned weak efficiency. The results are presented in table 3.

Autocorrelation function and Q test

The autocorrelation function and Q test are related methods to analyze if a given time series of data points is generated by an independent and identically distributed process. The Q test is applied using

Stock Markets			Foreign Exchange Markets		
Country	<i>GT</i>	Orden	Country	<i>GT</i>	Orden
Colombia	170.48	1	Colombia	170.48	1
Brazil	26.01	3	Brazil	30.61	4
Argentina	17.90	6	Argentina	158.95	2
Mexico	51.45	8	Mexico	41.81	3
United States	90.13	7	United States	NA	NA
Euro	NA	NA	Euro	11.57	6
England	45.68	5	England	3.71	8
Sweden	35.04	4	Sweden	9.28	7
Switzerland	22.31	2	Switzerland	24.78	5

Table 3: Geary Test: Left, Stocks Markets. Right, Foreigns Exchange markets

the following expression,

$$Q = n \sum_{\tau=1}^k \rho_{\tau}^2 \quad (18)$$

where ρ_{τ} are the correlation coefficients with time delay τ greater than zero,

$$\rho(\tau) = \frac{\sum_{t=1}^{n-\tau} (r_t - \bar{r})(r_{t+\tau} - \bar{r})}{\sum_{t=1}^n (r_t - \bar{r})}. \quad (19)$$

The serial correlation coefficient provides a measure of the relationship between the value of a random variable at time t and its value at time τ earlier [3]. In this work we estimate Q with τ equal to 20 because this is the mean number of working days in a month. The results are presented in table 4.

Stock Markets			Foreign Exchange Markets		
Country	<i>AF</i>	Orden	Country	<i>AF</i>	Orden
Colombia	87.06	1	Colombia	1394.55	1
Brazil	15.01	4	Brazil	166.46	3
Argentina	3.56	8	Argentina	228.34	2
Mexico	6.93	7	Mexico	13.42	6
United States	9.48	5	United States	NA	NA
Euro	NA	NA	Euro	15.58	5
England	72.13	2	England	15.64	4
Sweden	17.40	3	Sweden	12.10	8
Switzerland	7.07	6	Switzerland	12.19	7

Table 4: Q Test: Left, Stocks Markets. Right, Foreign Exchange markets

4 Summary and Conclusions

In each of the four tables, from table 1 to table 4, the last column indicates the order or position of the corresponding country (its corresponding market), from top, the less efficient, to bottom, the more efficient for the traditional weak efficiency criteria estimated with each one of the corresponding four methods. The results presented in the mentioned four tables, show that emerging and developed countries markets may show more or less weak efficiency depending on the method and on the origin of the information, stock or exchange market. However, some interesting results and tendencies can be obtained from the methods and from the markets themselves. Some of this results may be associated to well known characteristics of the market of a given country and-or to known economical and financial politics and legislation of the corresponding government.

In order to have a better view of the results we present in table 5 a synthesis of the results of the previous four tables: four methods for eight countries with information from the corresponding stock and exchange markets, for a total of 64 measures of weak efficiency. To analyze these interesting results, from the point of view of its potential useful economical and financial information, and the consistency of the different four methods as well, we developed a measure of concordance quantified in the concordance matrix CM . The elements of this matrix, CM_{ij} , are given by the following equation,

$$CM_{ij} = \sum_l \|X_{li} - X_{lj}\|, \quad (20)$$

where X_{lk} is the position of the country l with method k . The values obtained with equation (20), are an average measure of the concordance of the results of the different methods measuring the weak efficiency of the eight countries using the information of the corresponding exchange and stock markets. The elements CM_{ij} correspond to the mean value summing over the eight countries, of the difference between the position or order of each country (as presented in tables 1 to 5), corresponding to the methods i and j . The order of the country in each table is the relative weak efficiency of the eight countries accordingly to the corresponding method.

In table 6 we present the results of the concordance for the four methods to estimate the weak efficiency for stock and exchange markets. The concordance of two methods is normalized with a factor of 32 which corresponds to the maximum no concordance of two methods resulting from all the combinatorial alternatives of four methods and eight possible positions or order in a ordered list of eight countries from the less to the more efficient. Therefore, 0 corresponds to maximum concordance obtained only for a method with itself, and 1 for no concordance at all which indicates that the results of weak efficiency of two different methods are completely opposite.

In table 6 we observe qualitatively different results for the stock and exchange markets of the eight countries studied in this work. In the first case the results are very homogeneous with a bit larger concordance between the two entropy methods with the Run Test. In the case of the exchange market the

results are less homogeneous where the results of weak efficiency estimates with the Q test show a very bad concordance with the corresponding results of the relative entropy D , and the results of the Run Test show a strong concordance with the Pincus entropy $ApEn$.

These results do not permit to obtain definitive conclusions about weak efficiency and the capability of a particular method to measure it, more research is needed to determine a better understanding of the information that these methods are measuring and the corresponding relations with the weak efficiency as it is considered. However, from the scientific point of view of the information theory and statistical physics, and consistently with the results of this work, the entropy based methods may be more robust and fundamental but also strongly related with other methods, and may give useful financial and economical information from the empirical data of the corresponding markets that may help to improve the definition and the methods to measure weak efficiency and its relations with other characteristics of the stock and exchange markets.

Stock Markets					Foreign Exchange Markets				
Country	D	$ApEn$	GT	AF	Country	D	$ApEn$	GT	AF
Colombia	1	1	1	1	Colombia	2	2	1	1
Brazil	7	8	3	4	Brazil	6	3	4	3
Argentina	4	4	6	8	Argentina	1	1	2	2
Mexico	8	7	8	7	Mexico	4	4	3	6
United States	2	5	7	5	Euro	7	6	6	5
England	5	2	5	2	England	8	7	8	4
Sweden	3	6	4	3	Sweden	3	8	7	8
Switzerland	6	3	2	6	Switzerland	5	5	5	7

Table 5: Efficiency Markets for each method. Left: Stock Markets. Right: Foreign Exchange Markets

Stock Markets					Foreign Exchange Markets				
	D	$ApEn$	Run Test	Prueba Q		D	$ApEn$	Run Test	Prueba Q
D	0	0.4375	0.5	0.4375	D	0	0.3125	0.3125	0.625
$ApEn$	0.4375	0	0.5	0.4375	$ApEn$	0.3125	0	0.1875	0.3125
Run Test	0.5	0.5	0	0.4375	Run Test	0.3125	0.1875	0	0.375
Prueba Q	0.4375	0.4375	0.4375	0	Prueba Q	0.625	0.3125	0.375	0

Table 6: Concordance Matrix, left: Stock Markets. Right: Foreign Exchange Market

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4.1 Graphics

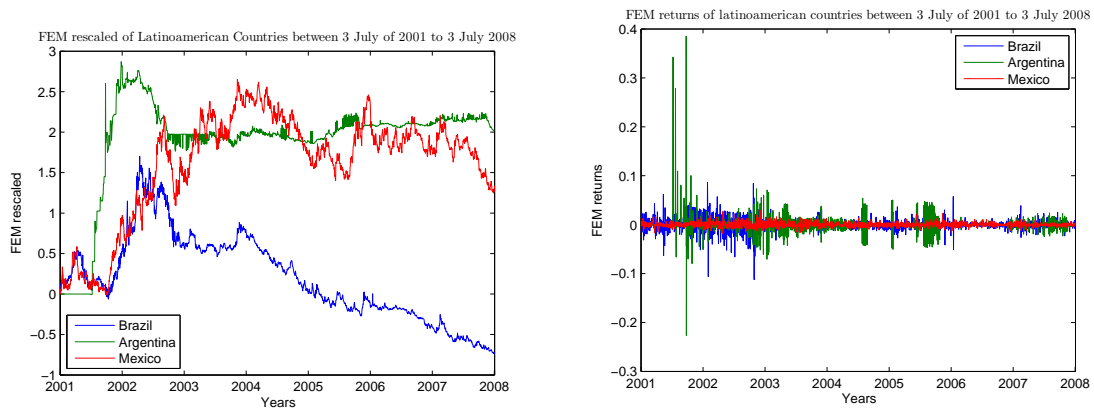


Figure 1: FEM and Returns of FEM for Latinoamerican Countries

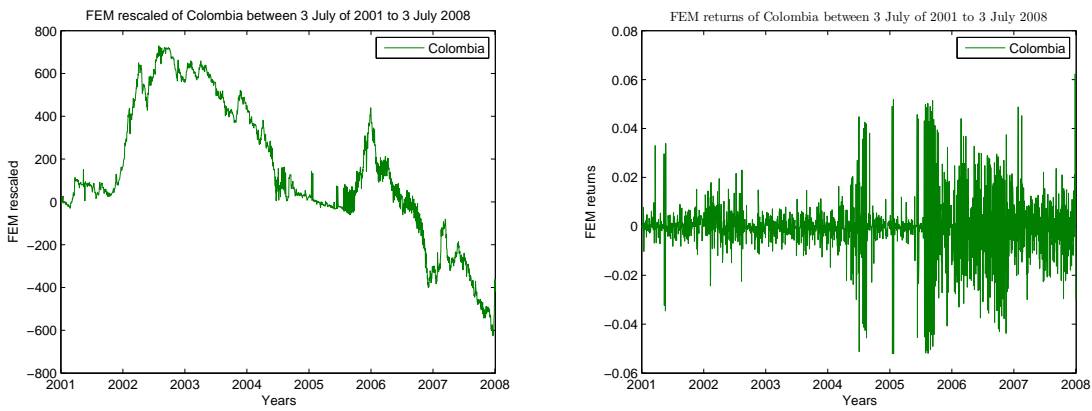


Figure 2: FEM and Returns for Colombia

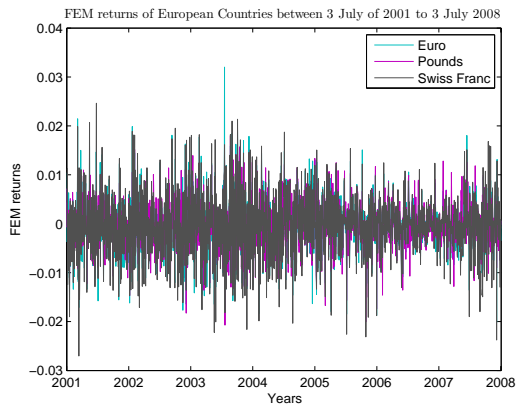
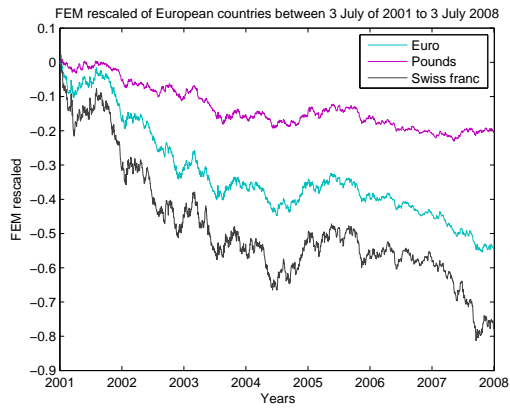


Figure 3: FEM and Returns of FEM for European Countries

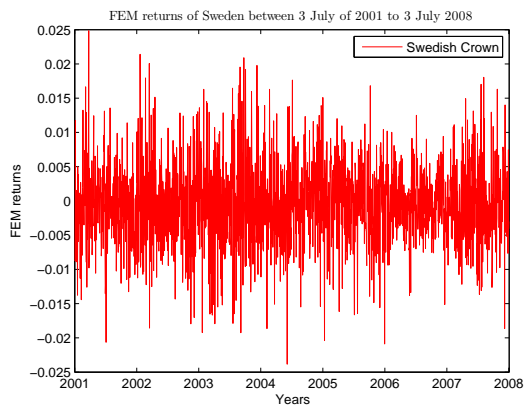
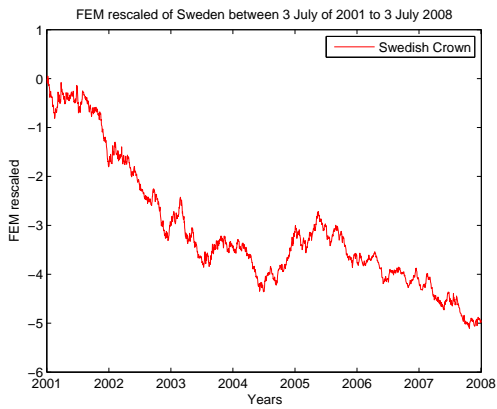


Figure 4: FEM and Returns for Sweden

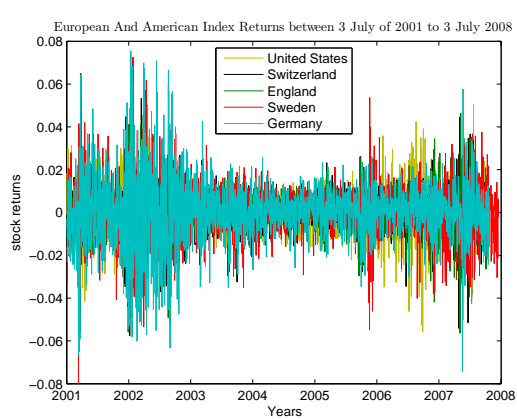
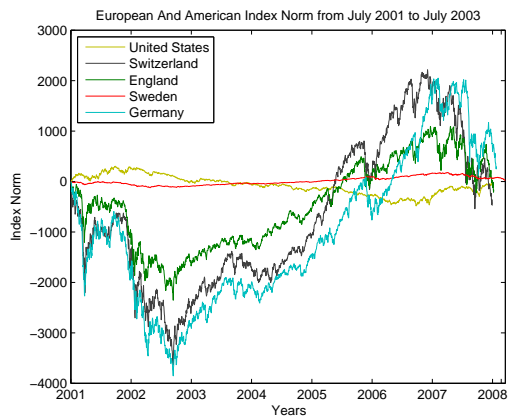


Figure 5: Index and Returns of European Countries

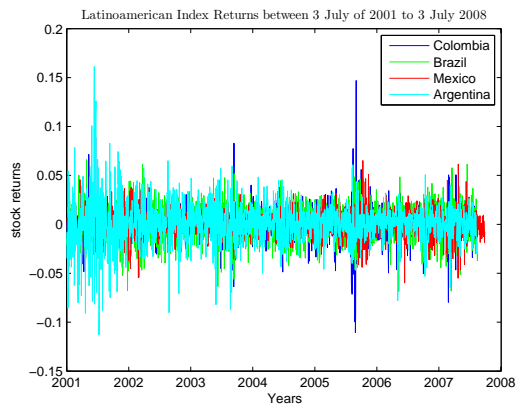
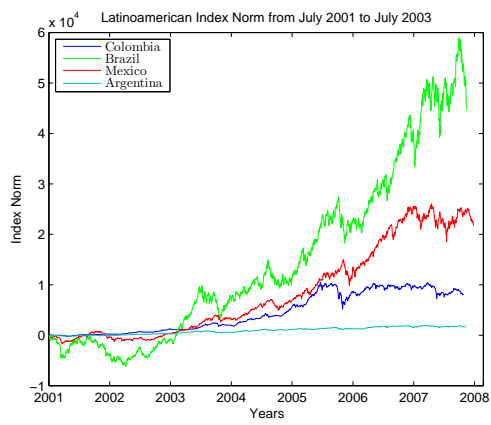


Figure 6: Index and Returns of Latinoamerican Countries