Differential Forms: A New Tool in Economics From Biological Models to Econophysics

Jürgen Mimkes [1]

Abstract

Investors would like to calculate profits in advance, "ex ante", like the path of a flying ball. But income can only be given "ex post", after the money is earned. The properties "ex ante" and "ex post" have never been investigated in economic theory by a mathematical tool. According to this paper the tool is given by exact and not exact differential forms. Integrals of inexact forms depend on the path of integration and may only be given "ex post", when the path of integration is known. The equivalence of monetary and production cycles leads to closed integrals of differential forms and a first and second law of economics, that deviates in many aspects from neoclassical theory. The most important result is entropy as the new production function, which enters production, trade, growth, optimization. Differential forms move economics close to thermodynamics and statistics. This new approach is called econophysics.

Introduction

Physics has always been a model for economics. Economists would like to calculate income and prices in advance like the path of a flying ball. But the flight of a ball may be calculated "ex ante", before the ball arrives. Stocks can only be valued "ex post", after trading. Economic theory of growth [1 – 3] is well aware of the problem of "ex ante" and "ex post". But so far this problem has never been investigated in a formal mathematical way. The same problem also appears in physics: In mechanics all functions can be calculated "ex ante" like the arrival of a flying ball. In contrast, thermodynamics does not even include time, nothing can be calculated "ex ante". Accordingly, we may expect thermodynamics and statisics to be a model science for economics [4]. This model function of physics is called econophysics [5] and has been applied mainly to finance. In this paper we will look into the properties "ex ante" and "ex post" starting from the biological models of economics.

Quesnay´s biological model

Economics is the science of production, distribution and consumption of commodities and services for people in societies. This corresponds to the supply of a living body with food and oxygen and early economists like Françoise Quesnay (1694 – 1774) have based the natural production cycle on the closed blood stream, fig 1:

[1] Physics Department, Paderborn University, D - 33096 Paderborn, Germany

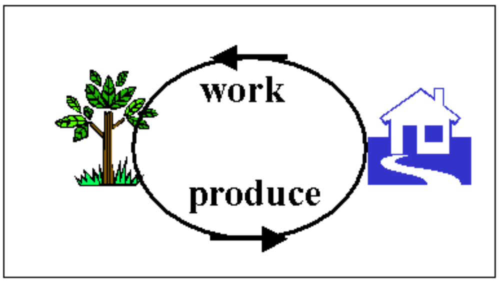

Fig. 1. Quesnay´s model of a natural production cycle of a simple economy. Work of laborers flows from households to the agriculture. In return produce flows as a reward from agriculture to households.

Labor flows from households to agriculture and consumption goods flow from agriculture to households. Consumption goods are the rewards of labor input.

Modern production cycles

The modern production cycle (δP) is more complex. Labor still flows from households to industry and consumption goods flow from industry to households. But the consumption goods are no more the reward for labor input. There is a second monetary cycle (δ Y). Labor is paid by wages and consumption goods of industry are paid by consumption costs of households. The monetary cycle (δ Y) measures the production cycle (δP), Eq.1:

$$\oint \delta Y = -\oint \delta P \qquad (1).$$

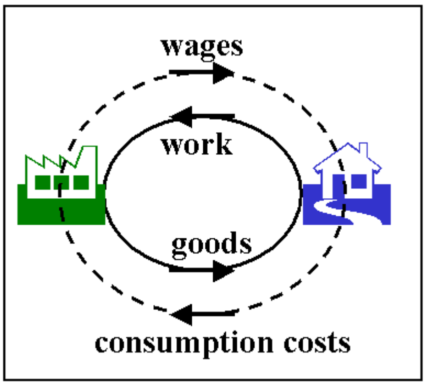

Fig. 2. The model of a modern, complex economy contains two cycles, a production cycle (solid line) and a monetary cycle (dashed line).

The law of economic survival

Neoclassical theory only considers the monetary cycle, which measures the production cycle. The monetary cycle may be split into two parts: income (Y) that flows from industry to households, and consumption costs (C) that flow from households to industry,

$$\oint \delta Y = Y - C = S > 0 \qquad (2).$$

Eq.2 is the law of survival for all economic systems, for households, companies, economies, societies – like for all living cells: the surplus (S) must be positive in order to secure survival and growth. The two biological models of economics, Eqs.1 and 2, are expressed by closed integrals, and we must now take a closer look at differential forms and their closed integrals..

Differential forms and their integrals

1. **Exact differential forms dY:** The closed integral of an exact differential form dY may be split into two integrals from (A) to (B) and from (B) to (A).

$$\oint dY = \int_A^B dY + \int_B^A dY = \int_A^B dY - \int_A^B dY \equiv 0 \qquad (3).$$

As the integral depends only on the limits (A) and (B), both integrals cancel. The closed integral of an exact differential form dY is always zero. An exact differential in two dimensions has the form $dY = a\,dx + b\,dy$, with $\partial b / \partial x = \partial a / \partial y = \partial^2 Y / \partial x\, \partial y$. The term ∂x denotes a partial differentiation. The function Y exists “ex ante”.

2. **Not exact differential forms δ Y:** The closed integral of a not exact differential form δ Y may be split into two integrals from (A) to (B) and from (B) to (A). As the integrals depend on the path, both integrals will not cancel [6 – 7]

$$\oint \delta Y = \int_A^B \delta Y + \int_B^A \delta Y = \int_A^B \delta Y - \int_A^B \delta Y \neq 0 \qquad (4).$$

The closed integral of a not exact differential form δY is never zero. A not exact differential in two dimensions, $\delta Y = a\,dx + b\,dy$ looks like an exact differential form, but the relation is now $\partial b/\partial x \neq \partial a/\partial y \neq \partial^2 Y / \partial x\, \partial y$. The value of Y can only be given “ex post”, when the path x of integration, the production process, is known. The function Y does not exist “ex ante”, each path (x) leads to a different function Y_x.

The first law of economics

The equivalence of production and monetary cycle, Eq.1, may be interpreted as the first law of economics: The differential forms δY and δP are equal up to a total differential form d K, as the closed integral of d K is zero. The resulting formula

$$\delta Y = dK - \delta P \qquad (5).$$

may be compared to the first law of thermodynamics, $\delta Q = dE - \delta W$. Income is given by (δY), production by (δP), (dK) is the capital of the economic system.

Economists are not familiar with the “first law of economics”, as they are not used to deal with differential forms. But economists agree with the results: 1. the monetary cycle measures the production cycle. 2: Income (δY) and production (δP) are characterized by “ex post”.

The second law of economics

A not exact differential (δ Y) may be turned into an exact differential (d F) by an integrating factor (1/ λ),

$$d F = \delta Y / \lambda \qquad (6).$$

F may be called production function. The law corresponds to the second law of thermodynamics, $d S = \delta Q / T$. The production function (F) is called entropy in physics. The meaning of the parameter (λ) depends on the system, in markets it is a mean price level, in societies the living standard, in economies the GDP per capita, in thermodynamics the mean energy per particle or temperature (T). Again economists are not familiar with the “second law of economics”, but agree with the results: 1. A production function (F) exists in every economic system. 2: The mean GDP per capita (λ) is an important economic parameter.

Econophysics

The laws (5) and (6) correspond exactly to the laws of thermodynamics. Indeed, not mechanics, but thermodynamics and statistics may be regarded as a model science for economics:

Y: income, costs, profits	Q: heat
P: production	W: work
F: production function	S: entropy
K: capital	E: energy
λ: mean capital level	T: mean energy level

This model character of thermodynamics may be called econophysics.

The entropy function of economic systems

The most important result is the equivalence of the production function F and entropy S. In the second law as well as in the Lagrange function the function F is given by entropy:

$$F \quad = \quad \ln \Omega \qquad (7).$$

Ω is the number of possibilities to place of N objects in K (price) classes:

$$\Omega \;=\; N! \,/\, (N_1 ! \ldots N_k !) \qquad (8)$$

For K = 2 we obtain the normal distribution. For large values of N the Stirling formula $\ln N! = N \ln N - N$ may be applied, which leads to

$$F(N_k) = \; N \ln N - \Sigma N_k \ln N_k \qquad (9)$$

The entropy function F depends only on the number N_k of items in K different classes of the economic system. The importace of entropy in economics has already been stated in the literature [8].

Entropy vs. Cobb Douglas function

Example: A Munich beer garden has N_1 permanent and N_2 temporary employees [9]. In fig. 3 the entropy function (9)

$$F(N_1) = (N_1 + N_2) \ln (N_1 + N_2) - N_1 \ln N_1 - N_2 \ln N_2 \quad (10)$$

and in fig. 4 the Cobb Douglas function [10]

$$U(N_1) = A\, N_1^{\alpha}\, N_2^{1-\alpha} \qquad (11)$$

are plotted versus N_1 in the range from 0 to 10. The parameter in both figures is N_2 in the range from 0 to 10.

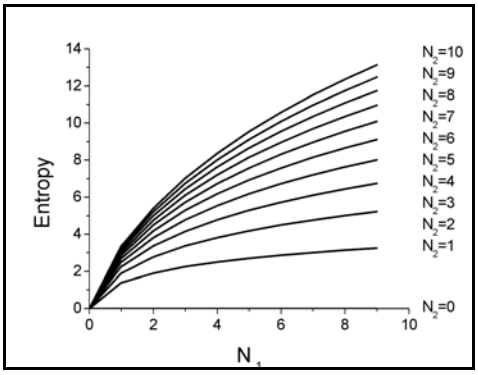

Fig. 3 Entropy $F(N_2) = (N_1 + N_1) \ln (N_1 + N_1) - N_1 \ln N_1 - N_2 \ln N_2$ plotted versus N_2 in the range from 0 to 10. The parameter is N_1 in the range from 0 to 10.

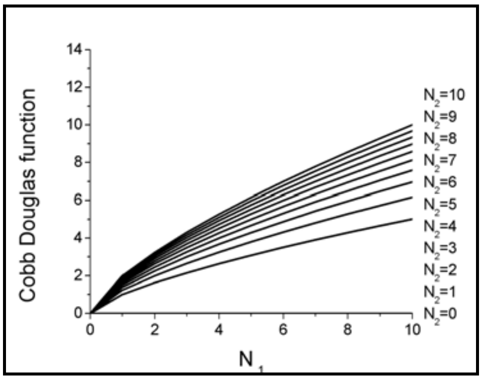

Fig. 4 The Cobb Douglas function $U(N_2) = A\ N_1^{\alpha}\ N_2^{1-\alpha}$ plotted versus N_2 in the range from 0 to 10. The parameter is N_1 in the range from 0 to 10. The parameters are chosen as A = 1 and α = 0,7.

All functions show marginal growth, as is observed in economics. The entropy function is larger than the Cobb Douglas function by a factor of about 1,4 for nearly all values of α. Entropy is clearly the better production function. In addition it is independent of any arbitrary elasticity parameter α and advancement of technology A.

Entropy is closely related to advancement of technology. Entropy is a measure of disorder. Producing a car from many different parts means ordering and reducing the disorder of parts (- dF), according to the first law, Eq.5: δP = dK - λ d F. Advancement of technology is again simplification of the production process, reduction of entropy of the production system. All busines is ordering, reducing entropy: Architekture is ordering of building plans. Companies are planing economic needs. Medicin is bringing a sick body into order. Law is ordering violations in society. Teaching is ordering young peoples brains. Science is ordering of natural phenomena.

Economic survival and the Carnot cycle

The law of economic survival, Eq.2, requires a permanently high income (Y) and low costs (C) in every economic system. This is made possible by differential forms and their path dependent integrals, which are interesting for engineers and economists. One can invest little in one way and receive much on the way back, as required by Eq.2. According to the second law of economics, δ Y = λ d F the integral Eq.2 may be carried out in the λ - F plane at constant λ and constant F in all economic systems,

$$\oint \delta Y = \oint \lambda dF = \lambda_2 \int_A^B dF - \lambda_1 \int_A^B dF = Y - C = S > 0 \qquad (12)$$

$$Y = \lambda_2\, \Delta F; \qquad C = \lambda_1\, \Delta F; \quad S = \Delta\lambda\, \Delta F > 0 \qquad (13)$$

Eq.12 is called a Carnot cycle. The integral is carried out at two different values of λ. Eq.13 is equivalent to the neoclassical approach, Y = F(K, N). With $\lambda_2 > \lambda_1$ the surplus S inis always positive. This is shown in fig. 5.

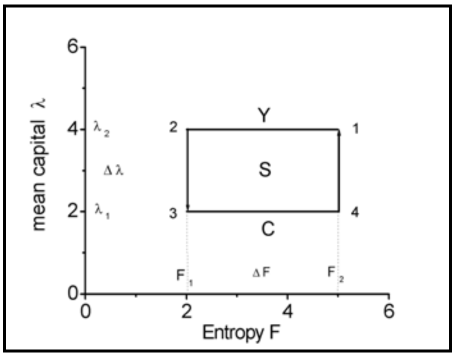

Fig. 5 shows the simple monetary cycle after Carnot in the $\lambda - F$ plane. Income corresponds to the area $Y = \lambda_2 \Delta F$, costs by the area $C = \lambda_1 \Delta F$. The surplus is the difference of the areas, $S = \Delta\lambda \Delta F$.

Example: Fig. 5 may be interpreted by the monetary cycle of the farm selling produce. $\Delta\lambda$ is the change in price levels bringing produce from the field to the market. $\Delta F > 0$: positive change of entropy = distribution (of money) to workers in the field, $\Delta F < 0$: negative change of entropy = collection (of money) from customers at markets.
1 $\rightarrow$ 2: Money collected from rich (λ_2) customers at the market, $Y = \lambda_2 \Delta F$.
2 $\rightarrow$ 3: Money transferred from market to fields, $\Delta F = 0$.
3 $\rightarrow$ 4: Money distributed to poor (λ_1) workers in the fields, $C = \lambda_1 \Delta F$.
4 $\rightarrow$ 1: Money brought by workers to the market, $\Delta F = 0$.

The Carnot Cycle of capital pumps

The Carnot cycle is a powerful mechanism in thermodynamics and economics:

1. a. A heat pump requires little energy to draw heat $Q_1 = T_1 \Delta S$ from a cold river and to deliver heat $Q_2 = T_2 \Delta S$ to a warm house with a high efficiency.

1. b. An import company pays low wages (C) for production in a poor country and receives a high income (Y) from sales in a rich country with a high efficiency. All monetary cycles of households, companies, states correspond to Eq.2 and the heat pump, and may be called capital pumps!

2. a. The motor is Carnot machine. The Carnot cycle of a motor runs in opposite direction of the heat pump. The motor transforms heat into consumed (kinetic) energy. The fuel is oil.

2. b. Production is a Carnot process. The production cycle runs in opposite direction of the monetary cycle. Production transforms resources (heat, material) into consumption goods. The fuel again is oil.
3. a. A refrigerator is another Carnot machine. It starts creating cold inside and warm outside immediately, after it is plugged to the wall. But we have to close the wall and separate inside and outside. The larger the difference, the higher the efficiency.
3. b. When a company starts producing, it will automatically create a richer and a poorer side. For this capital and labor have to be separated, or the company will not work. The economic gap between capital and labor will grow with time. The larger the difference, the higher the efficiency.

Economic growth

1. The Carnot cycle of a motor requires two temperatures, inside and outside. The heat that is created in every cycle, will dissipate to the inside and outside. The distribution of heat will influence the efficiency of the motor. If the inside gets hotter and hotter at constant outside temperature (due to a cooling system), the efficiency will grow quickly. If the outside runs hot (due to a failure of the cooling system), the motor stops.
2. The Carnot cycle creates two groups of people in farms, companies, business firms: farmer and laborers, owner and workers, capital and labor, first and third world, rich and poor, Y and C. Both groups together form the economic system. Accordingly, both groups will have to agree, how to divide the net output of each cycle. This is negotiated periodically by workers and employers, by unions and industry, by world trade conferences. In economic theory this is generally treated by game theory.

In a Carnot process the lower/cooler/poorer side (employees) will obtain the fraction (p) of the net output, the higher/ hotter/richer side (employer) will get the fracion (1 - p). If both groups reinvest their fraction $p\,(Y_2 - Y_1)$ and $(1-p)\,(Y_2 - Y_1)$, they will grow in time (t) with each cycle. We obtain two equations, for $C = Y_1(t)$ and $Y = Y_2(t)$:

$$d\,Y_1(t) = p\,(Y_2 - Y_1)\,d\,t \qquad (14)$$
$$d\,Y_2(t) = (1-p)\,(Y_2 - Y_1)\,d\,t \qquad (15).$$

For $p \neq ½$ the solution of this set of differential equations is given by:

$$Y_1(t) = Y_{10} + p\,[Y_{20} - Y_{10}]\,[\exp((1-2p)t) - 1]\,/(1-2p) \qquad (16)$$
$$Y_2(t) = Y_{20} + (1-p)\,[Y_{20} - Y_{10}]\,[\exp(1-2p)\,t) - 1]\,/(1-2p) \qquad (17).$$

For $p = ½$ the solution is given by

$$Y_1(t) = Y_{10} + ½\,[Y_{20} - Y_{10}]\,t \qquad (18).$$
$$Y_2(t) = Y_{20} + ½\,[Y_{20} - Y_{10}]\,t \qquad (19).$$

The equations may be applied to all interdependent systems, to workers and employers, unions and industry, or interdependent countries. The results are presented in fig. 6.

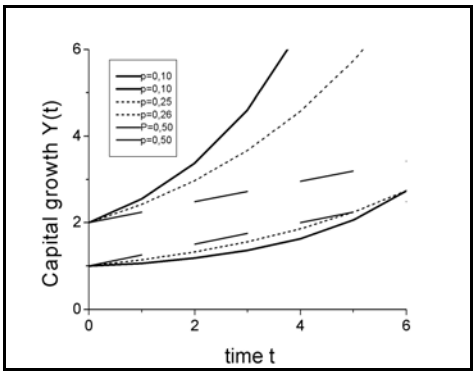

Fig. 6. Economic growth of a system of two parties (capital and labor) according to Eqs.14 to 19. One partner (Y_1) receives p percent of the surplus, the other partner (Y_2) receives (1-p) percent. For p = 0,10 we obtain high exponential growth, for p = 0,25 exponential growth is less. For p = 0,50 only linear growth is obtained.

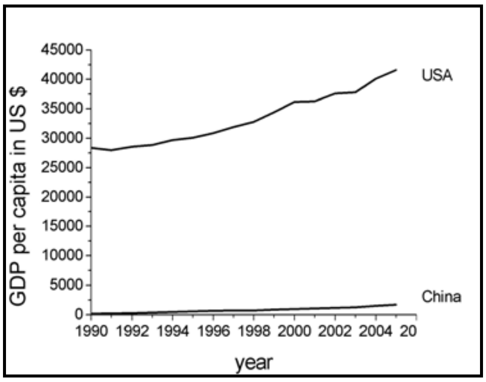

Fig. 7. Economic growth of a system of two parties (China and USA). The economic motor between China and western countries (USA) still is very efficient [11].

For p < 0,50 economic growth is exponential. Surprisingly a percentage of p = 10 % of the profit is in the long run more profitable for the worker than p = 25 %. The second surprise is the fair deal, 50:50. This distribution of profits leads only to linear growth and is the least attractive distribution between two partners. Fig 7 shows economic growth in China and USA between 1990 and 2005. The economic motor between these countries runs very well. Similar exponential growth has been observed in Germany and Japan after World War II, when trade with USA and other western countries led to exponential growth in both countries.

A high factor $p > 0,5$ leads to decreasing efficiency of the system. Y_2 and Y_1 are reaching a boundary. The economic motor runs hot and inefficient.
After Japan has acquired many production plants, the factor p has grown and the efficiency of the exports started to decrease. The GDP per capita of Japan now is trailing the slowly growing GDP per capita of USA.

Conclusion
Differential forms bring new ideas to economic theory, however, many methods of neoclassical theory and Lagrange optimization may be retained, if the Cobb Douglas function is replaced by the entropy of the economic system. Entropy makes calculation and understanding of economic processes more precise. Indeed, the concept of differential forms brings economics close to natural science.

Literature
1. Solow, A R. M., Contribution to the Theory of Economic Growth, The Quarterly Journal of Economics, 70 (Feb. 1956) 65-94

2. Valdes, B. D., Economic Growth: Theory, Empirics and Policy, Cheltenham, UK: Edward Elgar, (1999)

3. Barro, R. J. and Sala–i–Martin, X., *Economic Growth*, New York: McGraw-Hill, 1995. 2. Ed. MIT Press, Cambridge, Mass. (2004)

4. Mimkes, J., A Thermodynamic Formulation of Economics, in Econophysics and Sociophysics, B. K. Chakrabarti, A. Chakraborti, A. Chatterjee, (Eds.)
Wiley - VCH, Weinheim (2006) 1 - 33

5. Mantegna, R. N. and Stanley, H. E. *Introduction to Econophysics. Correlations and Complexity in Finance*. Cambridge University Press (2000)

6. Cartan, H., *Differential Forms*, Dover Publications; Translatio edition (2006)

7. Flanders, H., *Differential Forms with Applications to the Physical Sciences,*
Dover Publications Inc. (1990)

8. Georgescu-Roegen, N., *The entropy law and the economic process,* Cambridge, Mass. Harvard Univ. Press, (1974)

9. Bofinger, P., *Grundzüge der Volkswirtschaftslehre*, Pearson, Munich (2003)

10. Cobb, C. W. and , Douglas, P. H. : „A Theory of Production“ in American
Economic Review, Mar 28 Supplement, 18 (1928) 139-165

11. www.econstats.com/weo/V014.htm