Renormalization of random multiplicative processes and statistics of business-firm growth

Key words: Zipf law, growth rate, T-distribution, stable distribution, multiplicative stochastic process

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We introduce a simple mathematical model to explain the distribution of business-firm size[1] and growth rates[2]. We assume that the company consists of N independent sections whose time evolutions are given by the following set of equations:

$$\begin{cases} x_1(t+1) = b_1(t)x_1(t) + f_1(t) \\ x_2(t+1) = b_2(t)x_2(t) + f_2(t) \\ \vdots \\ x_N(t+1) = b_N(t)x_N(t) + f_N(t) \end{cases}$$

where $x_i(t)$ and $x_i(t+1)$ are sales of the i-th section in the year t and t+1, $b_i(t) \geq 0$ is the growth rate of this section, and $f_i(t)$ is an independent noise. Here, $b_i(t)$ and $f_i(t)$ are random variables characterized by given distributions. The whole sales of this firm, growth rate of the whole firm and the noise term for the whole firm is defined as

$$X_N(t) \equiv \sum_{k=1}^{N} x_k(t), \quad B_N(t) \equiv \sum_{k=1}^{N} b_k(t)x_k(t)/X_N(t), \quad F_N(t) \equiv \sum_{k=1}^{N} f_k(t),$$

which satisfy the following renormalized time evolution for the whole firm statistics:

$$X_N(t+1) = B_N(t)X_N(t) + F_N(t).$$

Our numerical and theoretical analyses derived many universal behaviors in the limit of very large N. The properties of firm growth statistics depend drastically on the conditions, $<b_i(t)>>1$, $<b_i(t)>=1$ or $<b_i(t)><1$. In the case of $<b_i(t)>=1$ the growth rate distribution converges a Student's T-distribution having power law tails independent of initial conditions. We also introduce renormalization in the time scale.

All results are compared with the real firm data which exhaustively covers practically all firms in Japan, about 1 million firms. It is shown that our simple model captures most of basic statistical properties.

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1 V. Pareto, Le Cours d'Économic Politique, Macmillan, London, 1897

2 M.H.R Stanley et al., Nature(London) 379, 804(1996)