NON-SELF AVERAGING OF A TWO-PERSON GAME ONLY WITH POSITIVE SPILLOVER: A NEW FORMULATION OF AVATAMSAKA DILEMMA PROCESS

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ABSTRACT. We have the Avatamsaka game [Aruka 2001] as a two-person game only with positive spillovers. In this game, selfishness would not be determined even if the agent selfishly adopted the strategy of defection. Individual selfishness could only be realized if the other agent cooperated. Any same-sized gain can be generated by either defection or cooperation. The sanction by defection as a reaction of the rival agent cannot necessarily reduce the selfishness of the rival. This game can be classified into a dependent game ([Akiyama and Aruka 2006]).

Aruka([Aruka 2002]) gave an idea to formulate an Avatamsaka game process in terms of Polya urn process. If we regarded an evolution of gain-ratio of each agent as a nonlinear function, the dynamics of gain addition could eventually give a limiting expected value. Here, agents shall then initially be motivated by the behaviors of the other agents, and in the event, agents' behavior could be independent from the others.

Now we introduce different spillovers, i.e., different pay-off matrices. Each agent may then be faced with a different pay-off matrix. A ball in the urn is interpreted with reference to the number of cooperation, while an urn a pay-off matrix. We apply Ewens' sampling formula to our urn process under this environment. In this case, we will have a similar result as in the classical case, because we have the *self averaging* on variances of the number of cooperation. We then apply Pitman's sampling formula to our urn process. Here the invariance of the random partition vectors under the properties of exchangeability and size-biased permutation does not hold in general. Incidentally, Pitman's sampling formula depends on the two parameter Poisson-Dirichlet distribution whose special case is just Ewens' formula. In the Ewens setting, it matters only one probability α on a new entry, on one hand. On the other hand, we can refer to an additional probability θ on an unknown type entry as will be argued in the Pitman formula.

More concretely, we will investigate the effects of differing pay-off sizes of playing a series of different games *coming out newly.* As [Aoki and Yoshikawa 2007] and [Aoki 2008] dealt with a product innovation and a process innovation, they criticized Lucas' representative method that Microsoft and small grocery store on the street face micro shocks drawn from the same unchanged probability distribution. In the light of [Aoki and Yoshikawa 2007], we may show the same argument in our Avatamsaka game with different pay-offs. In this setting, *innovations* occurring in *urns* may be regarded as *increases* of the number of cooperation in *urns* whose pay-offs are different.

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1. TWO-PERSON GAMES: FROM ONE PERSON TO DEPENDENT GAMES

In the Avatamsaka game [Aruka 2001], selfishness may not be determined even if the agent selfishly adopts the strategy of defection. Individual selfishness can only be realized if the other agent cooperates. Any certain gain from defection can never be assured by defection alone. The sanction by defection as a reaction of the rival agent cannot necessarily reduce the selfishness of the rival. In this game, there cannot be guaranteed *any explicit direct reciprocity.* The games may be characterized between the two poles of dependence and one person:

Definition 1. A game is called "Dependent Game", if the game belongs to a class of games in which the result for each agent will depend merely on the decisions of others, but not on his own decision. [Akiyama and Aruka 2006]

Definition 2. A game is called "One-person Game", if the game belongs to the complete independence (like in Robinson Crusoefs optimization problem), in which the decisions of others do not have any effect on the player's utility.

Avatansaka game([Aruka 2001]) argued here is one of the dependent games. While the Prisoners Dilemma [PD] as may be located in an intermediate point among the two edges in Figure 1. Since PD games sometimes are regarded as a so-called *bacillus coli* in social science experiments in the multi-agent based studies, it must be a good choice to compare our Avatamsaka game with a PD game.

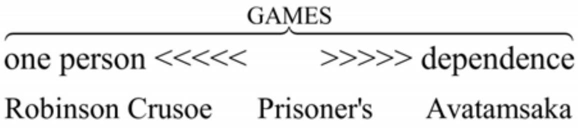

FIGURE 1

1.1. Avatamsaka game.

In order to clarify the characteristics of Avatamsaka game, we compare our Avatamsaka game with a class of Prisoner's Dilemmas. Here we take two numerical examples listed in Table 1 and 2.

	Player 2	
Player 1	Cooperation	Defection
Cooperation	(1, 1)	(0.25, 1)
Defection	(1, 0.25)	(0.25, 0.25)

TABLE 1. Avatamasaka

	Player 2	
Player 1	Cooperation	Defection
Cooperation	(1, 1)	(0.1, 0.9)
Defection	(0.9, 0.1)	(0.3, 0.3)

TABLE 2. PD

$$\text{Avatamsaka} \begin{bmatrix} R & S \\ T & P \end{bmatrix} = \begin{bmatrix} 1 & 0.25 \\ 1 & 0.25 \end{bmatrix} \quad \text{PD} \begin{bmatrix} R & S \\ T & P \end{bmatrix} = \begin{bmatrix} 1 & 0.3 \\ 0.9 & 0.1 \end{bmatrix}$$

In order to clarify the properties of the two games, we may use the letters R, S, T, P and the differences of $D_g = T - R,\ D_r = P - S$. Here we call

$$D_r = P - S$$

the Risk Aversion Dilemma;

$$D_g = T - R$$

the Gamble Intending Dilemma.

Firstly, we can specify the Avatamsaka game as follows:

$$D_g = T - R = 0 \tag{1}$$

$$D_r = P - S = 0 \tag{2}$$

On the other hand, a PD game is specified as follows:

$$D_g = T - R = 0.2 \tag{3}$$

$$D_r = P - S = 0.2 \tag{4}$$

Note that dilemmas may be generated unless "complementarily" for both players does hold.

We call

$$R - S = T - P$$

the **spillover**. In the Avatamsaka game, *spillovers* always are *positive*. So each player can then increase their rewards by the other player's strategy switching([Aruka 2001], 118). Regarding the Avatamsaka original papers, see [Aruka 2001], [Aruka 2002], and [Akiyama and Aruka 2006].

1.2. **Tanimoto's geometrical expression of two-person game.** We utilize the Tanimoto geometry to describe the characteristics of games. This method is proposed in [Tanimoto 2007a], [Tanimoto 2007b], and [Tanimoto and Sagara 2007].

The Tanimoto geometry is composed by the following equations:

$$P = 1 - 0.5 r_1 \cos\left(\frac{\pi}{4}\right) \tag{5}$$

$$R = 1 + 0.5 r_1 \cos\left(\frac{\pi}{4}\right) \tag{6}$$

$$S = 1 + r r_1 \cos\left(\frac{\pi}{4} + \theta\right) \tag{7}$$

$$T = 1 + r r_1 \sin\left(\frac{\pi}{4} + \theta\right) \tag{8}$$

$$\text{Here } r = \frac{r_2}{r_1}; r_1 = PS, r_2 = SM$$

By the use of this description, we can illustrate the key points R, S, T, P on the two-dimensional plane of Figure 2. Here the horizontal axis shows Player 1's pay-off while the vertical axis Player 2's one. An Avatamsaka game is shown in Figure 3 while a PD game in Figure 4.

We furthermore define the contour on the parametric plane (θ, r). We depict the contour of $T + P = R + S$ on this plane. A contour is shown in Figure 5. In this figure we have $T + P > R + S$ inside the contour and $T + P < R + S$ outside the contour. We put these relationships on Figure 6. A PD game is to be located on the vertical line $\theta = \frac{\pi}{2}$. In this point of view, a PD game may be regarded as a reference standard of any two-person game.

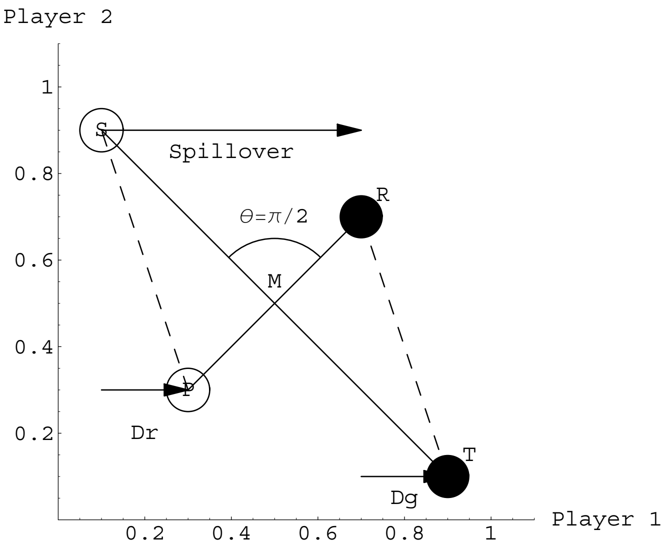

FIGURE 2. The Tanimoto geometry

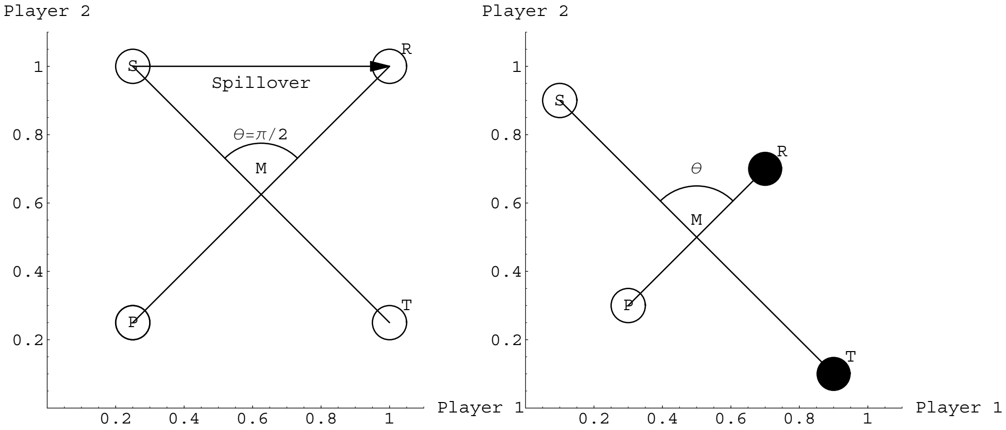

FIGURE 3. Avatamsaka

FIGURE 4. PD

2. THE AVATAMSAKA STOCHASTIC PROCESS UNDER A GIVEN PAY-OFF MATRIX

2.1. An application of Avatamsaka game to Polya urn stochastic process.

Now we apply a Polya urn stochastic process to our Avatamsaka game experiment

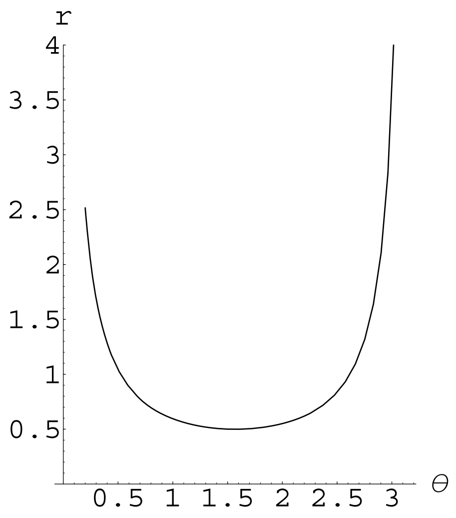

FIGURE 5

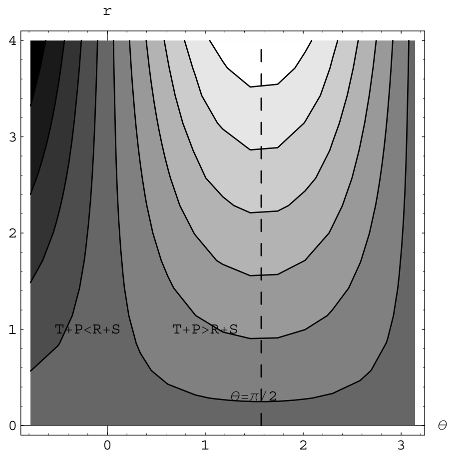

FIGURE 6

to predict an asymptotic structure of the game, according to [Aruka 2002]. This stochastic process was employed in the well-known book of Brian Arthur [Arthur 1994], strictly speaking in particular [Arthur-Ermoliev-Kaniovski 1987]. In order to apply Arthur's theorem, we impose the next assumptions.

Assumption 1. *We suppose there are a finite number of players, for instance, who join by each pair into our Avatamsaka Game.*

Assumption 2. *We identify the ratio of cooperation, or C-ratio for each player with the proportion of the total possible gains for each player.*

A classical Polya urn process. We interpret a red ball as a gain of Player who loves employing Defect Strategy in our two person game, while a white ball as a gain of Player who loves employing Cooperation Strategy. Player 1 can then continue to increase his gain if the proportion of his gain is kept higher than Player 2. The Polya original restriction never allows for Player to change his mind. A nonlinear inspection for Players may be taken for account to obtain a more realistic result.

Remark 1. We can have several applications of urn process to the Avatamsaka game, as we will show later. In the above we applied a classical Polya urn process with a given pay-off matrix to the Avatamsaka game. In this case, we may discover a kind of *averaging*, in a sense that players' behavior could be independent from the others. Fortunately, however, we have more candidates for doing this. We have Ewens' sampling formula: A K-dimensional Polya urn process with multiple pay-off matrices and possible new agents. Moreover, we have Pitman's sampling formula: two parameter Poisson-Dirichlet distribution. In the latter case, we may have a *non-averaging* property in a sense that the invariance of the random partition vectors under the properties of exchangeability and size-biased permutation does not hold in general. See [Pitman 2002].

2.2. **A gain-ratio in the total possible points.** Let X_i to be a gain-ratio in the total possible gains which Player i gains. Gain-ratio $= \frac{\text{Gain}}{\text{Total Gain}}$. The initial total potential of gains for each player is defined as $2N-1$. In the next period 2, the total maximal gains will grow by the number of players $2N$. In the period n, therefore, the gain-ratio for Player i at time n will be

$$X_i^n = \frac{b_i^n}{(2N-1)n}$$

Suppose such a sample space Ω that

$$\begin{cases} X_i : \Omega \to [0,1] \\ \omega \in \Omega \to X_i(\omega) \in [0,1]. \end{cases}$$

We can then have a probability $x = X_i(\omega)$ for a sample ω. If a sample implied an experimental result, the sample space Ω could make out *a psychological space* for each agent to give a next move. The probability x may depend on a random variable X_i.

2.3. **A nonlinear Polya urn dynamics.** The random vector of gain-ratios of period is:

$$X^n = (X_1^n, X_2^n, ..., X_{2N}^n)$$

The vector of the initial gains distribution is set:

$$b^1 = (b_1^1, b_2^1, ..., b_{2N}^1)$$

A probability of player i to earn **a point by means of Strategy** C in period n is set:

$$q_i^n(x)$$

One-dimensional dynamics of mapping then is defined as from X^n to X^{n+1}:

$$q_i(X_i)(\omega) \in [0,1]$$
$$q_i(x) : \Omega \to [0,1]$$

2.4. **The nonlinear evolution of X_i^n.**

$$\beta_i(X_i)(\omega) \in \{0,1\} \tag{9}$$

$$\beta_i(x) : \Omega \to \{0,1\} \tag{10}$$

The dynamics for addition of point(gain) is stated as:

$$b_i^{n+1} = b_i^n + (2N-1)\beta_i^n(x) \tag{11}$$

(12)

Here b_i^n is **an accumulated gain at time** n.

The nonlinear evolution of the proportion i.e., Arthur's dynamics :

$$X_i^{n+1} = X_i^n + \frac{1}{n+1}(\beta_i^n(x) - X_i^n) \tag{13}$$

Inspecting the expected value of X_i^n leads to the result:

Proposition 1. ([Aruka 2002]) *Players shall initially be motivated by the behaviors of the other players. Eventually, however, playersf behavior could be independent from the others.*

2.5. An another application to an urn stochastic modeling.

According to [Aoki and Yoshikawa 2007], we shall argue another urn modeling. Hence we employ the idea of *balanced urn* in the context of [Flajolet, Gabarro, and Pekari 2005].

Definition 3. We have two different colored balls: **white** and **red**. If we draw a red ball from the urn, we return a white balls and $b-a$ red balls into the urn. While, we return b red balls into the urn, if we draw a red ball from the urn. This type of ball replacement is expressed *the replacement matrix* R:

$$R = \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix}$$

The urn is called *balanced*, if it must be held
the first row's sum $= a + b - a = b$.
the second row's sum $= b$.
The matrix R then is *balanced triangular urn.*

Proposition 2. *The number of white balls in the balanced triangular urn model is non-self averaging.*

Proof. See ([Aoki and Yoshikawa 2007], 11-13) □

Just before discussing about the property of non-self averaging of strategies arrangement in a more general case, thus, we apply this modeling directly to our particular Avatamsaka game:

> If a player is faced to cooperation of a rival, the player can react by C-ratio of $\frac{a}{b-a}$. On the other hand, the player can react merely by defection when he is faced to defection of a rival.[1]

We then have the result that the game must be *non-self averaging* due to **Proposition 2**.[2]

3. THE KEY IDEAS FOR THE NEW ECONOMICS

3.1. The economics of master equation and the fluctuations.

The stochastic evolution of the state vector can be described in terms of the master equation as equivalent to the Chapman-Kolmogorov differential equation system. The master equation leads to bringing the aggregate dynamics, from which the Fokker-Planck equation could be derived. Thus we can explicitly argue the fluctuations in a dynamical system. These settings can indispensably be connected with the following key ideas making feasible the type classification of agents in the system, and to track the variations in cluster size. See [Aoki and Yoshikawa 2006] as been featured by *statistical physics* and *combinatorial stochastic processes.*

In Aoki's new economics, we have the exchangeable agents in the combinatorial stochastic process like in the urn process. The exchangeable agents come out by the use of random partition vector in the idea of statistical physics or population genetics. **The partition vector** provides us with the state information. We can thus argue the size-distribution of the components and their cluster dynamics with the exchangeable agents.

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[1] This statement needs such a kind of assumption as **Assumption 3** as will be given later in **Section 4.**

[2] We can add to a more general proposition: In non-balanced triangular urn models as depending on the values of parametewrs, no-self averaging emerges. See ([Aoki and Yoshikawa 2007],14).

Define an at most countable set, in which the probability density of transition from state i to state j is given respectively. In this setting, *dynamics of the heterogeneous interacting agents* gives the field where an agent can become another agent. It is also importantly noted that this way of thinking easily welcome the unknown agents.

A K-dimensional Polya distribution. We state a K-dimensional Polya distribution by the use of parameter θ. We then have **a Transition Rate**:

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \frac{n_i}{n} \frac{n_j + \theta_j}{n - 1 + \theta}$$

where

$$n = n_1 + \cdots + n_K; \theta_j > 0$$

and

$$\theta = \sum_j^K \theta_j$$

It then follows **a Jump Markov Process' Stationary State**:

$$\pi(\mathbf{n}) w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) w(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n}) \tag{14}$$

$$\pi(\mathbf{n}) = \frac{w(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})}{w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)} \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) \tag{15}$$

Hence we have **the Stationary Distribution**:

$$\pi(\mathbf{n}) = \frac{n!}{\theta^{[n]}} \prod_{i=1}^K \frac{\theta_i^{[n_i]}}{n_i!} \tag{16}$$

where

$$\theta^{[n]} = \theta(\theta + 1) \cdots (\theta + n - 1) \tag{17}$$

3.2. **A general urn process.** Suppose that **balls**(or agents) and **boxes**(or urns) are *both indistinguishable.* We then have a **Partition Vector**:

$$\mathbf{a} = (a_1, a_2, ..., a_n)$$

a_i is the number of boxes containing i balls. The number of balls is:

$$\sum_{i=1}^n i a_i = n$$

The number of categories is:

$$\sum_{i=1}^n a_i = K_n.$$

K_n is the number of occupied boxes.

The number of configurations then is

$$N(\mathbf{a}) = \frac{n!}{\prod_{j=1}^n (j!)^{a_j} a_j!} = \frac{n!}{(1!)^{a_1} (2!)^{a_2} \cdots (n!)^{a_n} a_1! a_2! \cdots a_n!}$$

A new type entry in an Urn process. Let $\mathbf{a}$ be a state vector. Suppose that one new type agent enters an empty box. We then have the equation:

$$w(\mathbf{a}, \mathbf{a} + \mathbf{e}_1) = \frac{\theta}{n+\theta} \tag{18}$$

Suppose also that an agent enters a cluster of size j. We then add one to size $j+1$ while reduce one from size j. We thus have:

$$w(\mathbf{a}, \mathbf{a} + \mathbf{e}_{j+1} - \mathbf{e}_j) = \frac{ja_j}{n+\theta} \tag{19}$$

On the contrary, suppose that an agent leaves a cluster of size j. We then add one to size $j-1$ while reduce one from size j. We thus have:

$$w(\mathbf{a}, \mathbf{a} - \mathbf{e}_j + \mathbf{e}_{j-1}) = \frac{ja_j}{n} \tag{20}$$

We then have **Ewens' Sampling Formula**

$$\pi(\mathbf{a}) = \frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} \left(\frac{\theta}{j}\right)^{a_j} \frac{1}{a_j!} \tag{21}$$

where

$$\sum_{j=1}^{n} ja_j = n; \sum_{j=1}^{n} a_j = K_n$$

The probability that the number of clusters is k. Let the probability that the number of clusters are k will be the sum of **a new comer who comes in a new cluster with probability** $\frac{n}{n+\theta}$ and **a new comer who comes in an existing cluster with probability** $1 - \frac{n}{n+\theta}$:.

$$q_{n,k} : = \Pr(K_n = k|n) \tag{22}$$

$$q_{n+1,k} = \frac{n}{n+\theta} q_{n,k} + \frac{\theta}{n+\theta} q_{n,k-1} \tag{23}$$

In this case, we have **the boundary conditions:**

$$q_{n,1} = \frac{(n-1)!}{\theta^{[n]}} \tag{24}$$

$$q_{n,n} = \frac{\theta^n}{\theta^{[n]}} \tag{25}$$

It then follows the solution:

$$q_{n,k} = \frac{\theta^k}{\theta^{[n]}} c(n,k) \tag{26}$$

where

$$\begin{aligned}
c(n+1,k) &= nc(n,k) + c(n,k-1) \\
\theta^{[n+1]} &= \sum_{m=0}^{n} s(n,m)\theta^m = \theta(\theta+1)(\theta+2)\cdots(\theta+n-1)(\theta+n) \\
&= \theta^{[n]}(\theta+n) = \theta\cdot\theta^{[n]} + n\cdot\theta^{[n]} \\
\theta\cdot\theta^{[n]} &= \sum_{m=0}^{n} s(n,m)\theta\cdot\theta^m = \sum_{m=0}^{n} s(n,m)\theta^{m+1} = \sum_{m-1=0}^{n} s(n,m-1)\theta^m = c(n,k-1) \\
n\theta^{[n]} &= n\sum_{m=0}^{n} s(n,m)\theta^m = n\sum_{m=0}^{n} s(n,m)\theta^m
\end{aligned}$$

The final equation is called signless **Stirling Number of the first kind**.

3.3. **Pitman's Chinese restaurant process.** We suppose that there are an infinite number of round tables in the Chinese restaurant that are labelled by an integer from 1 to n. The first customer, numbered 1, takes a seat at the table numbered 1. Suppose that the customers from No.1 to No.k in turn take their seats at their tables from No.1 to No. k. Here the c_j customers take their seats at the j-th table. See [Pitman 1995], and also see [Yamato and Shibuya 2000] and [Yamato and Shibuya 2003].

Now let that the new arrival comes out! The next arriving customer has **two options:** He or she can either take a seat at the k-th table by the probability

$$\frac{\theta + k\alpha}{\theta + n}$$

or at the table j, one of the remaining tables $j = 1, \cdots, k$ by the probability

$$\frac{c_j - \alpha}{\theta + n}$$

Here two parameters, θ and α are used. Thus we obtain the solution:

$$\frac{n!\theta^{[k:\alpha]}}{\theta[n]} \prod_{j=1}^{n} \left(\frac{(1-\alpha)^{[j-1]}}{j!} \right)^{c_j} \frac{1}{c_j!} \tag{27}$$

where

$$\begin{aligned}
\theta^{[j]} &= \theta(\theta+1)\cdots(\theta+j-1) \\
\theta^{[j:\alpha]} &= \theta(\theta+\alpha)\cdots(\theta+(j-1)\alpha)
\end{aligned}$$

Ewens' sampling formula [Ewens 1972] gives the invariance of the random partition vectors under the properties of exchangeability and size-biased permutation. The Ewens sampling formula is the case with one parameter, a special case of two parameter Poisson-Dirichlet distributions:

$$\frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} \left(\frac{\theta}{j} \right)^{a_j} \frac{1}{a_j!}$$

In the case of the two-parameter Poisson-Dirichlet model, we will be faced with a *non-averaging* system in the limit.

4. AN APPLICATION OF THE TWO-PARAMETER POISSON-DIRICHLET MODEL TO AVATAMSAKA

4.1. **Non-self averaging.** [Aoki and Yoshikawa 2006] has shown that the two-parameter Poisson-Dirichlet models are qualitatively different from the one-parameter version. As shown in ([Aoki and Yoshikawa 2007],6). An additional parameter could then generate *non-self averaging*, even if we had a situation of *self averaging* in the one-parameter model. We call an urn state *self averaging*, if the number of balls of each urn could eventually be convergent in average the following sense. We cite from ([Aoki and Yoshikawa 2007],4):

Definition 4. "Non-self-averaging" means that a size-dependent (i.e.,"extensive" in physics) random variable X of the model has the coefficient of variation that does not converge to zero as model size goes to infinity. The coefficient of variation $C.V.$ of an extensive random variable, X, defined by

$$C.V.(X) = \frac{\sqrt{variance(X)}}{mean(X)}$$

is normally expected to converge to zero as model size (e.g. the number of economic agents) goes to infinity. In this case, the model is said to be "self-averaging."

4.2. **Aoki's application.** In an application of the two-parameter Poisson-Dirichlet model to an economic system, [Aoki and Yoshikawa 2007] gives a growing economy of multi-sectors where the waves of *innovations* are arriving *stochastically.* That is to say,

> an innovation, when it occurs, either raises productivity of one of the existing sectors, or creates a new sector. Thus, the number of sectors is not given, but increases over time.
>
> By the time n-th innovation occurs, the total of K_n sectors are formed in the economy where in the i-th sector has experienced n_i innovations ($i = 1, 2, ..., K_n$). By definition, the following equality holds:
>
> $$n_1 + n_2 + \cdots + n_k = n \quad (*)$$
>
> when $K_n = k$. If n-th innovation creates a new sector (sector k), then $n_k = 1$.

Between different spillovers. Now we turn to our *Avatamsaka game.* As we have already seen, the Avatamsaka game was charcterized by the positive spillovers for both players. Either greater or smaller spillovers than a given one *all* preserve Avatamsaka characteristics, of course. We illustrate a typical change of spillover size in Figure 7. We can measure a difference of any two spillovers by

$$\frac{\eta_1}{\eta_2} = \frac{SR}{S'R'}$$

Incidentally, we note that we can get a greater spillover structure in a PD game by simply extending the PR line.

Given a particular spillover, the expected total gain in the urn will increase, if the number of cooperation in the game urn increases. It is natural that **more cooperative players** could bring about a higher average sized gain in the given box. This coincides with the previous assumption (Assumption 2).

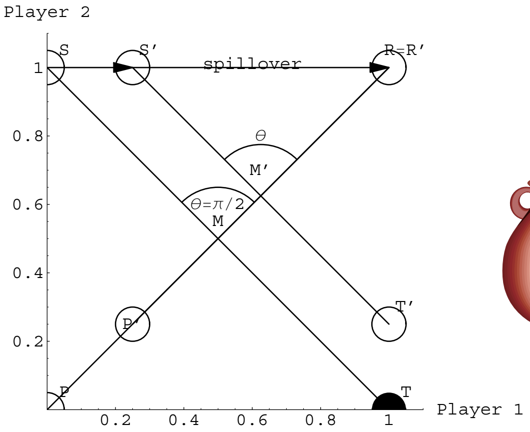

FIGURE 7. spillover

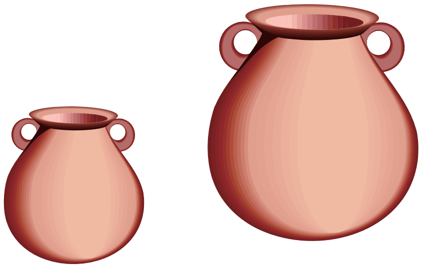

FIGURE 8. Various spillovers

4.3. **An application to Avatamsaka game.** Suppose there are a multiple different urns with various different payoff matrices, each of which has each own size of spillover. A different spillover in our Avatamsaka game may change the inclinations of reaction of players. But these inclinations are not necessarily symmetrical. An urn with a greater spillover might sometimes be more attractive for a cooperative player in the high level of cooperation, because the players could earn their greater gains in size. While a cooperative player in a state of the low level of cooperation might feel attractive to enter a urn with a smaller spillover in a much higher level of cooperation. But he can change his mind to be a defective player. There may thus be various players' plans for their strategies. Any player decisively depends on his state to which he remains or enter, while an urn, i.e., a pay-off matrix occurs stochastically.

Thus we suppose that any urn must be random for players, i.e., the other way around, namely, any player must be random for urns. In our Avatamsaka game, we may rather neglect the size effects due to different spillovers. So we need not presume any specifically particular transition mechanism *ex ante*.

Now we apply this relationship to our Avatamsaka game space. That is to say, we shall replace "innovations" with "increases of the number of cooperation" in our game.

> By the time n-th cooperation occurs, the total of K_n payoff urns are formed in the whole game space where in the i-th payoff urn has experienced n_i coopeations ($i = 1, 2, ..., K_n$). By definition, the following equality holds:

$$n_1 + n_2 + \cdots + n_k = n \tag{28}$$

> when $K_n = k$. If n-th cooperation creates a new payoff matrix (urn k), then $n_k = 1$.

It is noted that

$$n = \sum_j j a_j(n) \tag{29}$$

So we suppose that there are a finite number of urns into which various types of pay-off matrices are embedded: $1, \cdots, K$.

In this environment, we can have n inventions to increase the number of cooperation x_i. In other words, the number of cooperation in urn i may grow due to a stochastic multiple inventions occurring in this urn.

Assumption 3. *Cooperation accelerates cooperation, i.e., the larger the number of coopeartions, the larger the number of cooperation and/or the total gain in the urn will be.*

Due to an Avatamsaka property, we moreover impose another assumption.

Assumption 4. *Player can compare the situations between the urns given him by normalizing his own gain.*

We then assume a particular type growth:

$$x_i = \eta_i \gamma^{n_i}, \ \eta_i > 0, \gamma > 1, \text{for } i = 1, ..., k. \tag{30}$$

First of all, the part γ^{n_i} reflects **Assumption 3**.

As shown in the above subsection, η_i indicates an element of set of different spillovers:

$$\mathrm{E} = (\eta_1, \cdots, \eta_n).$$

$a_j(n)$ is the number of urns where j inventions have occurred. Here $a_j(n)$ is an element of the partition vector $a(n)$. K_n can then be expressed as

$$K_n = \sum_j^n a_j(n). \tag{31}$$

Expanding the exponential $\exp(n_i \ln \gamma)$ and rounding the remaining terms except for the first two, we obtain the next approximation:

$$\gamma^{n_i} \approx 1 + \ln(\gamma) n_i$$

Hence it follows:

$$x_i = \eta_i + \eta_i \ln(\gamma) n_i. \tag{32}$$

Due to **Assumption 4**, we normalize x_i by the use of η_i.

$$\tilde{x_i} = \frac{x_i}{\eta_i}$$

Here we then define:

$$X_n = \sum_i^{K_n} \tilde{x_i}. \tag{33}$$

This shows the aggregate behavior of cooperation dynamics, i.e., a cluster urn dynamics. Thus, from these equations (28)-(32) in the above, we obtain

$$X_n \approx K_n + \beta \sum_j^n j a_j(n). \tag{34}$$

FIGURE 9. A stochastic interaction

where $\beta = \ln(\gamma) > 0$. It then turns out that X_n depends on how cooperation occurs.

4.4. **A result.** We have just transformed our Avatamsaka game form into the same equation (34) in the essentially same context as [Aoki and Yoshikawa 2007]. Thus, by the same reasoning, we could conclude the same results. Hence we have the following propositions.[3]

Proposition 3. *In the two-parameter Poisson-Dirichlet model, the aggregate cooperation behavior X_n is non-self averaging.*

Proof. See ([Aoki and Yoshikawa 2007], 6-10). □

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[3] ([Aoki and Yoshikawa 2007],6) argued the economic meaning of *non-self-averaging* as follows:

> The notion of *non-self-averaging* is important because non-self averaging models are sample dependent, and some degree of impreciseness or dispersion remains about the time trajectories even when the number of economic agents go to infinity. This implies that focus on the mean path behavior of macroeconomic variables is not justified. It, in turn, means that sophisticated optimization exercises which provide us with information on the means have little value.

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