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Theory of Financial Markets: Handout

Information Transmission under Strategic Behavior

Due to the assumption of the random arrival of traders and the predetermined size of transactions, the model of Glosten and Milgrom did not allow for *strategic* behavior of traders. The second benchmark model of the microstructure literature (Kyle, 1985) incorporates the strategic choices of insiders' transaction volume and shows that this added flexibility for informed agents' market operations does not prevent revelation of their private information.

Assumptions:

- trading is again organized in a competitive system of specialists or market makers (the zero profit condition applies), but now they match demand and supply in a batch-clearing settings: *all customers submit market orders to the specialists who then determine the market price at which they are willing to satisfy these orders.* As a consequence, there is only one price and the bid-ask spread collapses to zero;
- the volume of market orders is arbitrary, and becomes a strategic choice variable,
- market orders of liquidity traders are formalized by random draws from a Normal distribution, i.e. liquidity traders' excess demand is $\mu \sim N(0, \sigma_\mu^2)$,
- the fundamental value is stochastic and is also determined by a random draw from a Normal distribution: $p_f \sim N(\overline{p}, \sigma_p^2)$,
- there is one monopolistic, risk-neutral insider who knows the realization p_f, whereas all other agents only know its distribution.

1 The Insider's Choice Problem

The insider maximizes the expected profit from his informational advantage. Denoting by p the market price determined by the market makers and by x excess demand ($\gtreqless 0$) of the insider, his profit turns out to be:

$$\pi = (p_f - p) \cdot x. \tag{1}$$

Presumably, the market price will depend on overall excess demand, $\mu + x$, so that the insider faces a trade-off: *the higher his excess demand or supply, $|x|$, the higher the profit for a given price p.* However, a high order would exert upward or downward pressure on p, so that the price would move closer to p_f and the wedge between the fundamental value and the price in (1) would decline.

In order to solve this problem, the insider has to know the pricing strategy of the market makers. From the zero-profit condition it follows that:

$$p = E\left[p_f | x + \mu\right], \tag{2}$$

since market makers condition their expectations on overall excess demand as their source of information. Because of the existence of the insider with his superior information, high excess demand (or supply) would, *ceteris paribus*, suggest a positive (negative) realization of p_f. However, the exact (Bayesian) way to infer information on p_f from $x + \mu$ depends in turn on the order strategy of the insider which is still unknown.

We, therefore, encounter a problem of strategic interaction in which each party's reaction function (order strategy and pricing strategy) depends on the strategy of the opponent. Usually one would like to determine a Nash equilibrium of this strategic game in which each reaction function is the best response to the strategy of the opponent. Since there might be multiple Nash equilibria a typical approach consists in trying certain types of solutions and checking whether consistent best responses exist for these candidate solutions.

Since the simplest candidate solution is a linear one, we might try a pricing strategy of market makers as follows:

$$p = P(x + \mu) = \overline{p} + \lambda \cdot (x + \mu), \tag{3}$$

with λ: price adjustment speed. This has the intuitively plausible property that the price will be determined above (below) the mean value $\overline{p}$ of the distribution of p_f if overall excess demand is positive (negative).

With this candidate strategy of the market makers, the insider can easily determine his profit maximizing order strategy:

$$\pi = x \cdot (p_f - \overline{p} - \lambda \cdot (x + \mu)). \tag{4}$$

Therefore,

$$E[\pi] = E[x \cdot (p_f - \overline{p} - \lambda \cdot (x + \mu))] = x \cdot (p_f - \overline{p} - \lambda x). \tag{5}$$

$$\frac{dE[\pi]}{dx} = p_f - \overline{p} - 2\lambda x = 0,$$

$$\Rightarrow x = \frac{1}{2\lambda}(p_f - \overline{p}). \tag{6}$$

We find that the (i) the insider's order will be positive (negative) if $p_f > \overline{p}$ ($p_f < \overline{p}$) holds, (ii) the absolute magnitude of the order depends inversely on the market makers' price adjustment speed λ.

2 The Market Makers' Inference Problem

Consistency of the solution to the insider's reaction function implies that the market makers' best response to (6) is their price adjustment function (3) which has been used as the starting point of the above derivation. In our setting this means that with this price setting they correctly incorporate the informational content of their observation, $x + \mu$, into prices (and, hence, the zero profit condition applies):

$$E\left[p_f | x + \mu\right] = \overline{p} + \lambda \cdot (x + \mu). \tag{7}$$

We, therefore, have to show that (7) is the result of Bayesian information processing!

Denote excess demand by θ:

$$\theta = x + \mu = \frac{1}{2\lambda}(p_f - \overline{p}) + \mu = \beta \cdot (p_f - \overline{p}) + \mu, \tag{8}$$

with $\beta \equiv \dfrac{1}{2\lambda}$.

Equation (8) shows that θ is a noisy signal of p_f as it includes p_f (via the insider's order), as well as some deterministic (β, $\overline{p}$) and stochastic quantities (μ). Note that:

$$E[\theta] = \beta \cdot (p_f - \overline{p}). \tag{9}$$

Consider the following simple transformation:

$$z \equiv \frac{\theta}{\beta} + \overline{p} = \frac{\beta \cdot (p_f - \overline{p}) + \mu}{\beta} + \overline{p} = p_f + \frac{\mu}{\beta}, \tag{10}$$

with $E[z] = p_f$.

Via some elementary calculations we, therefore, can obtain a new observation , z, whose expectation is p_f!

Note also that:

$$Var[z] = \frac{1}{\beta^2}\sigma_\mu^2 = 4\lambda^2\sigma_\mu^2 \tag{11}$$

Since μ is Normally distributed, z also follows a Normal distribution: $z \sim N(p_f, \frac{1}{\beta^2}\sigma_\mu^2)$. We, thus, have a noisy signal of the realization p_f whose variance (i.e. its precision) is also known.

The task is now, to apply Bayesian updating to this setting: *a Normally distributed ex-ante probability, $N(\overline{p}, \sigma_p^2)$, and a Normally distributed signal, z, have to be combined to arrive at the ex-post probability distribution of p_f on the base of the additional information from the observed transaction volume.*

For curious readers, a full treatment of the Bayesian learning problem for Normally distributed random variables is available in the Appendix. Here we only summarize the results:

- if both the ex-ante probabilities and the signal follow a Normal distribution, the ex-post probability is also characterized by a Normal distribution (this conservation property does not hold for most other distributions),
- for a Normally distributed prior distribution and a Normally distributed signal, the mean value of the ex-post distribution is a weighted average of the ex-ante expectation and the signal. The weights of both components are their *degrees of precision* (precision = 1/variance). Since these weights usually do not sum up to one, one has to divide the resulting expression by the sum of these weights.
- the precision of the ex-post distribution is the sum of the degree of precision of the ex-ante distribution and the degree of precision of the signal. The variance of the ex-post distribution is, then, obtained as the inverse of the ex-post precision.

Application of these results gives:

$$E\left[p_f | x + \mu\right] = E\left[p_f | z\right] = \frac{\dfrac{\overline{p}}{\sigma_p^2} + \dfrac{z}{4\lambda^2\sigma_\mu^2}}{\dfrac{1}{\sigma_p^2} + \dfrac{1}{4\lambda^2\sigma_\mu^2}}, \tag{12}$$

$$Var\left[p_f|z\right] = \left(\frac{1}{\sigma_p^2} + \frac{1}{4\lambda^2\sigma_\mu^2}\right)^{-1}. \tag{13}$$

Now let us see whether the ex-post probability distribution squares with our starting point, the candidate for the pricing strategy, eq. (7):

$$\overline{p} + \lambda \cdot (x + \mu) = \frac{\dfrac{\overline{p}}{\sigma_p^2} + \dfrac{z}{4\lambda^2\sigma_\mu^2}}{\dfrac{1}{\sigma_p^2} + \dfrac{1}{4\lambda^2\sigma_\mu^2}}. \tag{14}$$

This consistency requirement is only met if we choose an appropriate value for the "free" parameter λ. Solving for this parameter yields[1]:

$$\lambda = \frac{1}{2}\frac{\sigma_p}{\sigma_\mu}. \tag{15}$$

We, therefore, have found one set of reaction functions which is consistent with a Nash equilibrium of this strategic interaction problem:

Market makers:	$p = \overline{p} + \frac{1}{2}\frac{\sigma_p}{\sigma_\mu}(x + \mu)$
Insider:	$x = \frac{\sigma_\mu}{\sigma_p}(p_f - \overline{p})$

Remarks:

- note that both the insider's order strategy and the market makers' pricing strategy are fully specified, and only depend on the parameters for the two sources of uncertainty $(p_f,\, \mu)$,
- the order size of the insiders is the higher, the higher σ_μ (better chance to hide among liquidity traders) and the lower σ_p (if ex-ante precision is high, excess demand will have less of a price impact),
- the market makers' price sensitivity is the higher the higher σ_p (as transaction volume contains more information in this case) and the lower σ_μ (since, then, the signal becomes more precise),
- we can now also compute the expected profits of the monopolistic insider:

$$\begin{aligned} E[\pi] &= x \cdot (p_f - \overline{p} - \lambda x) \\ &= \frac{1}{2\lambda}(p_f - \overline{p})(p_f - \overline{p} - \lambda\frac{1}{2\lambda}(p_f - \overline{p})) \\ &= \frac{1}{2}\frac{\sigma_\mu}{\sigma_p}(p_f - \overline{p})^2 = \frac{1}{2}\sigma_\mu\sigma_p > 0. \end{aligned}$$

[1] What we have done here is known as the method of undetermined coefficients and is widely used in economic models with rational expectations (note that our agents also have rational expectations about their opponent reactions in a Nash equilibrium): *one starts with a functional form with undetermined coefficients for the solution of a model and determines these coefficients from the consistency requirements imposed by rational expectations.*

3 Revelation of Information

Note that the variance of the signal can be simplified:

$$4\lambda^2\sigma_\mu^2 = \frac{\sigma_p^2}{\sigma_\mu^2}\sigma_\mu^2 = \sigma_p^2 \tag{16}$$

Hence, the conditional expectation (= market price) and variance are:

$$\begin{aligned} p &= E\left[p_f|z\right] = \frac{\frac{\overline{p}}{\sigma_p^2} + \frac{z}{\sigma_p^2}}{\frac{1}{\sigma_p^2} + \frac{1}{\sigma_p^2}} \\ &= \frac{1}{2}(\overline{p} + z), \end{aligned} \tag{17}$$

$$Var\left[p_f|z\right] = \frac{1}{2}\sigma_p^2. \tag{18}$$

The expectation of the market price is:

$$E[p] = \frac{1}{2}(\overline{p} + E[z]) = \frac{1}{2}(\overline{p} + p_f). \tag{19}$$

Therefore, *on average*, the market price will move into the direction of the "true" fundamental value, and the uncertainty (variance) is reduced by a factor 2 after one trading round: *the insider will on average reveal half of his information through his order.*

4 Multiple Trading Rounds

A sequence of trading rounds provides noisy signals z_1, z_2, ... The market price in the first round is

$$p_1 = \frac{1}{2}(\overline{p} + z_1), \tag{20}$$

as shown above. Prior to the second trading round, the ex-ante probabilities are replaced by the ex-post probabilities, i.e. the distribution $p_f(z_1) \sim N(\frac{1}{2}(\overline{p} + z_1), \frac{1}{2}\sigma_p^2)$ is used instead of the original $p_f \sim N(\overline{p}, \sigma_p^2)$.

One could now work through the entire model on the base of the new probability distribution $p_f(z_1)$. However, in order to see the consequences of multiple trading rounds one only needs to modify (17) and (18) by using the mean and variance of $p_f(z_1)$. We, then, obtain for the second round:

$$\begin{aligned} p_2 &= E\left[p_f|z_1, z_2\right] = \frac{1}{2}(p_1 + z_2) \\ &= \frac{1}{2}\left(\frac{1}{2}(\overline{p} + z_1) + z_2\right), \end{aligned} \tag{21}$$

with

$$E[p_2] = \frac{1}{4}\overline{p} + \frac{3}{4}p_f, \tag{22}$$

since $E\left[z_1\right] = E\left[z_2\right] = p_f$, and

$$Var\left[p_f|z_1, z_2\right] = \left(\frac{1}{\frac{1}{2}\sigma_p^2} + \frac{1}{\frac{1}{2}\sigma_p^2}\right)^{-1} = \frac{1}{4}\sigma_p^2. \tag{23}$$

It should now be straightforward to see what happens in periods 2, 3, ... In general, we find:

$$\begin{aligned} p_n &= E\left[p_f | z_1, z_2, ..., z_n\right] \\ &= \frac{1}{2} z_n + \frac{1}{4} z_{n-1} + ... + \frac{1}{2^n}(\overline{p} + z_1), \end{aligned} \tag{24}$$

$$E[p_n] = \frac{1}{2^n}\overline{p} + \frac{2^n - 1}{2^n} p_f, \tag{25}$$

$$Var\left[p_f | z_1, z_2, ..., z_n\right] = \frac{1}{2^n}\sigma_p^2. \tag{26}$$

We see that:

$$E[p_n] \longrightarrow p_f \text{ for } n \to \infty, \tag{27}$$

and

$$Var\left[p_f | z_1, z_2, ..., z_n\right] \longrightarrow 0 \text{ for } n \to \infty. \tag{28}$$

I.e., asymptotically all private information is revealed through repeated trading. In finite periods, the market price contains all publicly available information, so that the market is semi-strong form efficient. Fig. 1 illustrates the shifts of the ex-post distribution over multiple trading rounds.

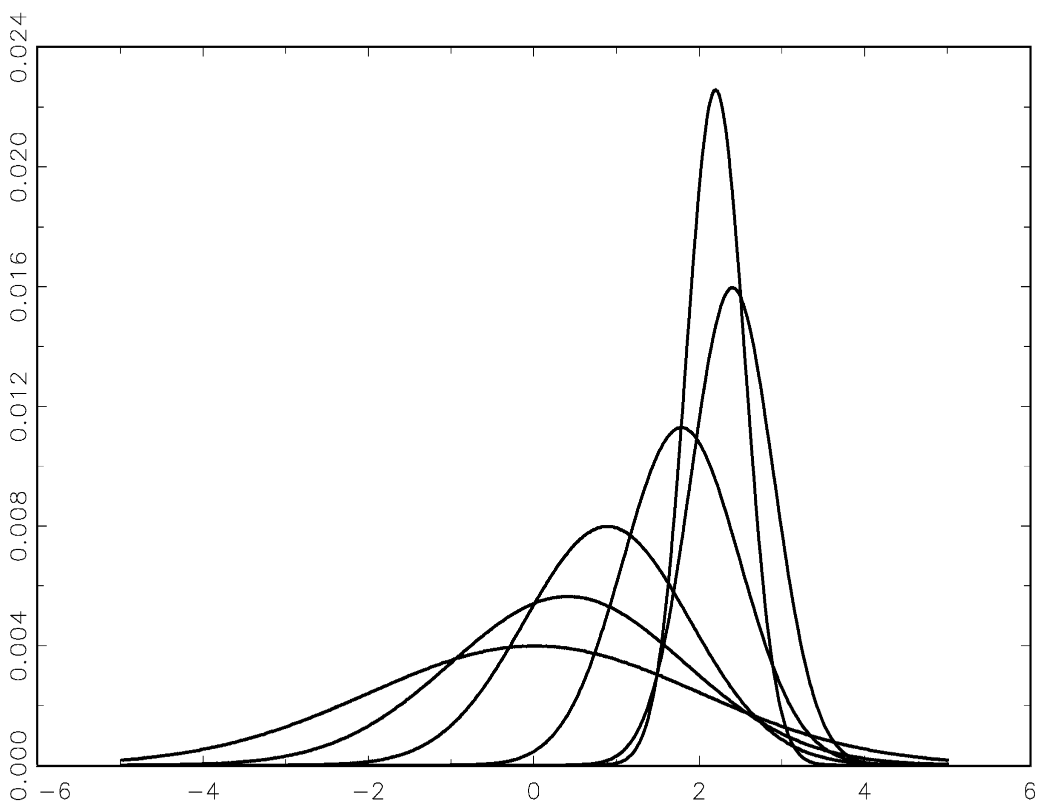

Figure 1: An example for the time development of the posteriori density in the Kyle (1985) model at $t = 1, 2, ...6$ with $p_0 = 0$, $p_f = 2$, and $\sigma_p = 2$.

5 Strategic Timing

The above multi-period analysis assumed that the insider would choose strategies maximizing the expected profit for the current period. Kyle (1985) also studies an explicit intertemporal optimization problem which has more complicated equilibrium strategies. In particular, the reaction coefficients, λ and β, *change* in each period depending on previous market outcomes. With intertemporal optimization the insider would be better able to exploit the fluctuations of liquidity traders' orders, in that he would increase his reaction after a "misleading" realization of μ_t etc.

Admati and Pfleiderer (1989) study a model in which both insiders and liquidity traders have the flexibility to strategically choose their timing of transactions. Liquidity traders would like to concentrate their trading in one period if that leads to lower transaction costs by signaling a mass of uninformed trading volume to market makers (lower bid-ask spread). Of course, insiders would like to join into this cluster which depending on the relative size of both groups might rather lead to increasing costs. But if the disadvantages dominates, no clustering equilibrium would exist and liquidity traders would prefer more dispersed trading activity.

An important consequence of these considerations is that one would expect a negative relationship between transaction volume and the size of the bid-ask-spread. However, this implication of microstructure theory is typically not confirmed by the data: *on the intradaily level, pronounced asymmetries of trading volume are accompanied by similar intradaily movements of spreads.*

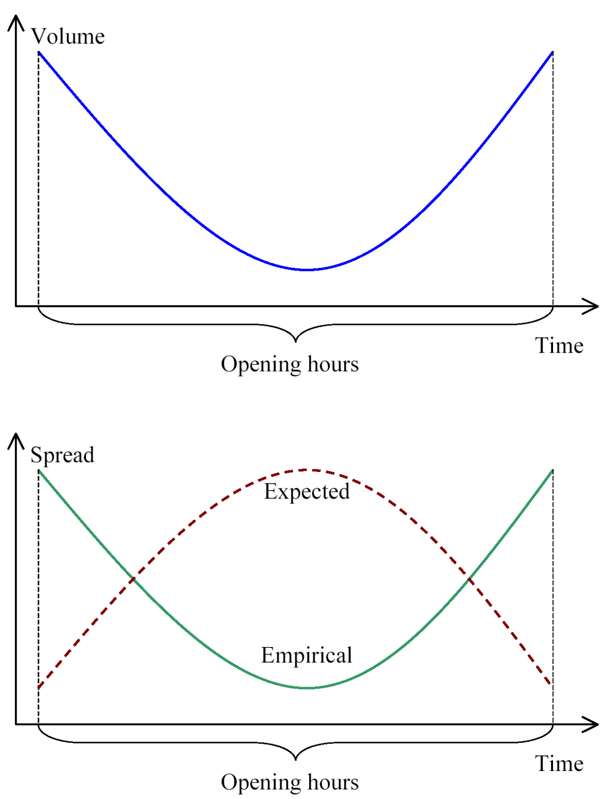

Figure 2: Typical (schematic) intradaily variation of volume and bid-ask-spread.

Reference:

1. Admati, A. and P. Pfleiderer (1988), *A Theory of Intraday Patterns: Volume and Price Variability*, Review of Financial Studies 1, pp. 3-40.

2. Kyle, A. (1985), *Continuous Auctions and Insider Trading*, Econometrica 53, pp. 1315-1336.

3. O'Hara, M. (1995), *Market Microstructure Theory*, Oxford, ch. 4.

Appendix:

Bayesian Learning with Normally Distributed Random Variables[2]

Bayesian updating for continuously distributed random variables:

Denote by:

y: event,
x: observation,
$g(y)$: prior density,
$f(x|y)$: conditional density of x,
$\int_y f(x|y)g(y)dy$: unconditional density for observations x.

Applying Bayes's rule to continuous random variables yields the posterior density:

$$g(y|x) = \frac{g(y)f(x|y)}{\int_y f(x|y)g(y)dy}. \tag{A1}$$

For Normally distributed random variables we find:

Assume the prior density follows a Normal distribution with mean μ and variance σ_y^2:

$$g(y) = \frac{1}{\sigma_y\sqrt{2\pi}}e^{\left[-\frac{(y-\mu)^2}{2\sigma_y^2}\right]}. \tag{A2}$$

The observation x is a noisy signal for y, with Normally distributed noise with variance σ_x^2:

$$f(x|y) = \frac{1}{\sigma_x\sqrt{2\pi}}e^{\left[-\frac{(x-y)^2}{2\sigma_x^2}\right]}, \tag{A3}$$

$\rightarrow$ posteriori density $g(y|x)$ is obtained as:

$$g(y|x) = \frac{h(x,y)}{\int_{-\infty}^{+\infty} h(x,y)dy}, \tag{A4}$$

with

$$\begin{aligned} h(x,y) &= \frac{1}{2\pi\sigma_y\sigma_x}e^{\left\{-\frac{1}{2}\left(\frac{1}{\sigma_y^2}(y-\mu)^2+\frac{1}{\sigma_x^2}(x-y)^2\right)\right\}} \\ &= \frac{1}{2\pi\sigma_y\sigma_x}e^{\left\{-\frac{(y-\mu)^2}{2\sigma_y^2}-\frac{(x-y)^2}{2\sigma_x^2}\right\}}. \end{aligned} \tag{A5}$$

[2] This Appendix provides a derivation of results used in the main test, but this material is not relevant for the exam to this course.

One solves:

$$\begin{aligned}
\frac{(y-\mu)^2}{2\sigma_y^2}+\frac{(x-y)^2}{2\sigma_x^2} &= \frac{1}{2}\left\{\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}\right)y^2-2\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)y+\frac{\mu^2}{\sigma_y^2}+\frac{x^2}{\sigma_x^2}\right\} \\
&= \frac{\Sigma}{2}\left[y^2-2\frac{y}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]+\frac{1}{2}\left(\frac{\mu^2}{\sigma_y^2}+\frac{x^2}{\sigma_x^2}\right) \\
&= \frac{\Sigma}{2}\left[y-\frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]^2-\frac{1}{2\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)^2+\frac{1}{2}\left(\frac{\mu^2}{\sigma_y^2}+\frac{x^2}{\sigma_x^2}\right) \\
&= \frac{\Sigma}{2}\left[y-\frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]^2+\frac{(\mu-x)^2}{2\left(\sigma_y^2+\sigma_x^2\right)}, \qquad \text{(A6)}
\end{aligned}$$

with:

$$\Sigma=\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}=\frac{\sigma_x^2+\sigma_y^2}{\sigma_x^2\sigma_y^2}. \qquad \text{(A7)}$$

Hence:

$$\begin{aligned}
h(x,y) &= \frac{1}{2\pi\sigma_y\sigma_x}e^{\left\{-\frac{\Sigma}{2}\left[y-\frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]^2-\frac{(\mu-x)^2}{2\left(\sigma_y^2+\sigma_x^2\right)}\right\}} \\
&= \left[\frac{1}{2\pi\left(\sigma_y^2+\sigma_x^2\right)}\right]^{\frac{1}{2}}e^{\left\{-\frac{(\mu-x)^2}{2\left(\sigma_y^2+\sigma_x^2\right)}\right\}}\cdot\underbrace{\left[\frac{\sigma_x^2+\sigma_y^2}{2\pi\sigma_x^2\sigma_y^2}\right]^{\frac{1}{2}}e^{\left\{-\frac{\Sigma}{2}\left[y-\frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]^2\right\}}}_{\substack{\text{density of a Normal distribution}\\ \text{with mean } \frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right) \text{ and}\\ \text{variance } \frac{1}{\Sigma}}} \qquad \text{(A8)}
\end{aligned}$$

Note that

$$\int_{-\infty}^{+\infty}\left[\frac{\sigma_x^2+\sigma_y^2}{2\pi\sigma_x^2\sigma_y^2}\right]^{\frac{1}{2}}e^{\left\{-\frac{\Sigma}{2}\left[y-\frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]^2\right\}}dy = 1, \qquad \text{(A9)}$$

while $\left[\frac{1}{2\pi\left(\sigma_y^2+\sigma_x^2\right)}\right]^{\frac{1}{2}}e^{\left\{-\frac{(\mu-x)^2}{2\left(\sigma_y^2+\sigma_x^2\right)}\right\}}$ does not depend on y and under the same integral can be treated as constant, thus:

$$\int_{-\infty}^{+\infty}h(x,y)dy=\left[\frac{1}{2\pi\left(\sigma_y^2+\sigma_x^2\right)}\right]^{\frac{1}{2}}e^{\left\{-\frac{(\mu-x)^2}{2\left(\sigma_y^2+\sigma_x^2\right)}\right\}}. \qquad \text{(A10)}$$

It follows that the posteriori density can be derived as:

$$
\begin{aligned}
g(y|x) &= \frac{h(x,y)}{\displaystyle\int_{-\infty}^{+\infty} h(x,y)dy} \\
&= \frac{\left[\frac{1}{2\pi\left(\sigma_y^2+\sigma_x^2\right)}\right]^{\frac{1}{2}} e^{\left\{-\frac{(\mu-x)^2}{2\left(\sigma_y^2+\sigma_x^2\right)}\right\}} \cdot \left[\frac{\sigma_x^2+\sigma_y^2}{2\pi\sigma_x^2\sigma_y^2}\right]^{\frac{1}{2}} e^{\left\{-\frac{\Sigma}{2}\left[y-\frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]^2\right\}}}{\left[\frac{1}{2\pi\left(\sigma_y^2+\sigma_x^2\right)}\right]^{\frac{1}{2}} e^{\left\{-\frac{(\mu-x)^2}{2\left(\sigma_y^2+\sigma_x^2\right)}\right\}}} \\
&= \left[\frac{\sigma_x^2+\sigma_y^2}{2\pi\sigma_x^2\sigma_y^2}\right]^{\frac{1}{2}} e^{\left\{-\frac{\Sigma}{2}\left[y-\frac{1}{\Sigma}\left(\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}\right)\right]^2\right\}}. \qquad \text{(A11)}
\end{aligned}
$$

$\rightarrow$ ex-post probabilities follow a Normal distribution with mean:

$$E[y|x] = \frac{\frac{\mu}{\sigma_y^2}+\frac{x}{\sigma_x^2}}{\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}}$$

and variance:

$$Var[y|x] = \left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}\right)^{-1}.$$

Reference:

Berger. J. O., *Statistical Decision Theory and Bayesian Analysis*, 2nd ed., Springer, 1993.